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$U_q(sl_2)$ AT FOURTH ROOT OF UNITY

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Contents

1 Introduction

This paper originated in a double motivation.

In physics, we have strong reasons to believe that $SL_q(2,\mathbb{C})$ for a primitive third root of unity is fundamentally related with the fermion structure [1]. A precise implementation of this idea would open extremely interesting perspectives. In the noncommutative geometry version of the standard model of elementary particles [2][3], replacement of the present phenomenological "spectral triple" by a fundamental Ansatz is expected to yield a scenario for a new "supersymmetric" standard model and a calculation of the fermion masses.

In mathematics, after Alain Connes axiomatic construction of spin-manifold [1], one is tempted by the project of an analogous theory of "generalized supermanifolds" (which we call "medusae"). This area, inevitable inasmuch as these objects exist (classical supermanifolds, gauge degrees of freedom of field theories) is mysterious and difficult, because the corresponding algebras are no longer subalgebras of C^*-algebras. In fact these are non semi-simple algebras, semi-simplicity arising only after quotienting by their nilradical. The projected survey of "medusae" lacks the two guiding features which have led Connes to the axiomatics of spin-manifold: the guidance through physics (hardly procured by the present supersymmetric standard model -in our opinion too ugly to be fundamental!) and (non semi-simplicity pending) the absence of Hilbert space techniques (presumably leading to abysses like "Hilbert spaces with indefinite metric", a situation already found but poor! ly investigated in quantum field theory).

We believe that the (finite dimensional) $U_q(sl_2)$ at roots of unity are models which hopefully yield features suggesting axioms for "medusae". A feature found in the two examples of $U_q(sl_2)$ for $q^3 = 1$ [4] and the present H_1^i is the striking (apparently new) fact that the trace of the adjoint representation (in the quantum group sense [6]) has the nilradical in its kernel. This fact, in combination with an appropriate $*$-operation entails that semi-simplicity is synonymous with "positivity" (as is the case for the transversal degrees of freedom of quantum electrodynamics in the Lorentz gauge, eliminated by the requirement of a "strictly positive" Hilbert space, see e.g. [5])

We conjecture that semi-simplicity and positivity are synonymous for all roots of unity.

Apart from this feature, our paper displays various aspects of the quantum groups H_N^i, including a complete description of their algebra automorphisms and Hopf $*$-structures.

2 $U_q(sl_2)$ at fourth root of unity

$U_q(sl_2)$ is the algebra defined by the symbols K, K^{-1}, E, F and the relations

$$\begin{cases} KE = q^2 EK, \\ KF = q^{-2} FK, \\ [E,F] = \frac{K-K^{-1}}{q-q^{-1}}. \end{cases} \tag{1}$$

It has a Hopf algebra structure defined by

$$Comultiplication \begin{cases} \Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}, \\ \Delta(K) = \mathbf{K}\otimes\mathbf{K}, \\ \Delta(K^{-1}) = \mathbf{K^{-1}}\otimes\mathbf{K^{-1}}, \\ \Delta(E) = \mathbf{E}\otimes\mathbf{1} + \mathbf{K}\otimes\mathbf{E}, \\ \Delta(F) = \mathbf{F}\otimes\mathbf{K^{-1}} + \mathbf{1}\otimes\mathbf{F}, \end{cases} \tag{2}$$

$$Counity \begin{cases} \varepsilon(\mathbf{1}) = \varepsilon(\mathbf{K}) = \varepsilon(\mathbf{K^{-1}}) = \mathbf{1}, \\ \varepsilon(E) = \varepsilon(\mathbf{F}) = \mathbf{0}, \end{cases} \tag{3}$$

$$\text{Antipode}\begin{cases} S(1)=\mathbf{1}\otimes\mathbf{1},\\ S(K)=\mathbf{K^{-1}},\\ S(K^{-1})=\mathbf{K},\\ S(E)=-\mathbf{K^{-1}E},\\ S(F)=-\mathbf{FK}.\end{cases} \tag{4}$$

When we consider the special case $q=i$, the relations 1 become

$$\begin{cases} KE=-EK,\\ KF=-FK,\\ [E,F]=\frac{K-K^{-1}}{2i}.\end{cases} \tag{5}$$

The Casimir operator is defined as follows.

$$C=FE+\frac{K-K^{-1}}{4i}. \tag{6}$$

Lemma 2.1 *E^2, F^2 and K^2 belong to the center of $U_i(sl_2)$.*

Proof:

K^2 commutes with K, E and F since it anticommutes with E, F. E^2 commutes with K since E and K anticommute. E^2 also commutes with F,

$$[E^2,F]=E[E,F]+[E,F]E=E\frac{K-K^{-1}}{2i}+\frac{K-K^{-1}}{2i}E=0.$$

Similarly, F^2 commutes with K since E and K anticommute. F^2 commutes with E

$$[E,F^2]=F[E,F]+[E,F]F=F\frac{K-K^{-1}}{2i}+\frac{K-K^{-1}}{2i}F=0.$$

□

Definition 2.1 *We define $H_N^i=U_i(sl_2)/I_N$, with I_N the ideal of $U_i(sl_2)$ generated by $E^2, F^2, K^{2N}-\mathbf{1}$.*

Proposition 2.1 *1) H_N^i is a Hopf algebra with the Hopf structure 2-4 and the Casimir operator 6.*
2) H_N^i has a PWB-base $\{F^pK^nE^q\}_{p,q=0,1;n=0,\dots,N}$. Thus it has dimension $8N$.

Proof:

In fact, we have in $U_i(sl_2)$

$$\begin{aligned}&\Delta(E^2)=E^2\otimes\mathbf{1}+\mathbf{K^2}\otimes\mathbf{E^2},\\ &\Delta(F^2)=F^2\otimes K^{-2}+\mathbf{1}\otimes\mathbf{F^2},\\ &\Delta(K^{2N}-\mathbf{1})=\mathbf{K^{2N}}\otimes(\mathbf{K^{2N}}-\mathbf{1})+(\mathbf{K^{2N}}-\mathbf{1})\otimes\mathbf{1},\end{aligned} \tag{7}$$

$$\begin{aligned}&\varepsilon(E^2)=0, && S(E^2)=-K^{-2}E^2,\\ &\varepsilon(F^2)=0, && S(F^2)=-F^2K^2,\\ &\varepsilon(K^{2N}-\mathbf{1})=\mathbf{0}, && S(K^{2N}-\mathbf{1})=\mathbf{K^{-2N}}(\mathbf{K^{2N}}-\mathbf{1}),\end{aligned} \tag{8}$$

1) The relations are immediately checked using multiplicativity of Δ and ε, as well as antimultiplicativity of S. These relations imply that I_N is a Hopf ideal. Indeed one has from 7 the inclusion $\Delta(I_N)\subset I_N\otimes H_N^i+H_N^i\otimes I_N$, and I_N contains the elements $\varepsilon(E^2)$, $\varepsilon(F^2)$, $\varepsilon(K^{2N}-\mathbf{1})$, $S(E^2)$, $S(F^2)$, $S(K^{2N}-\mathbf{1})$ owing to 8.

2) Follows from the multiplication table below. □

We now introduce a convenient alternative parametrization of the N-dimensional algebra $\mathbf{K}$ generated by K. Owing to $K^{2N}-\mathbf{1}$, $\mathbf{K}$ is the group algebra of the finite abelian group $\mathbb{Z}/2N\mathbb{Z}$. Using harmonic analysis on this group, we replace the basis $\{K^n\}_{n=0,..,2N-1}$ by the Fourier transformed basis $\{e_n\}_{n=0,..,2N-1}$, leading to a simpler description.

Lemma 2.2 *Setting a complex number* $u = e^{\frac{2i\pi}{2N}}$*, the* $*$*-symmetric elements :*

$$e_k = e_k^* = \frac{1}{2N} \sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{kj} K^j \qquad k \in \mathbb{Z}/2N\mathbb{Z}, \tag{9}$$

with reversion formulae

$$K^j = \sum_{k \in \mathbb{Z}/2N\mathbb{Z}} u^{-kj} e_k \qquad j \in \mathbb{Z}/2N\mathbb{Z}, \tag{10}$$

yield a basis of K*. It has the following properties,*

$$\begin{array}{ll} 1) & \sum_{k \in \mathbb{Z}/2N\mathbb{Z}} e_k = \mathbf{1}, \\ 2) & e_k e_m = \delta_{km} e_k, \\ 3) & K e_k = u^{-k} e_k, \\ 4) & K^{-1} e_k = u^k e_k, \\ 5) & E e_k = e_{k+N} E, \\ 6) & e_k F = F e_{k+N}, \\ 7) & C = EF + \frac{1}{4i} \sum_{k \in \mathbb{Z}/2N\mathbb{Z}} (u^k - u^{-k}) e_k. \end{array} \qquad k \in \mathbb{Z}/2N\mathbb{Z} \tag{11}$$

Proof:

The equivalence of 9 and 10 stems from the obvious fact that $\sum_{k \in \mathbb{Z}/2N\mathbb{Z}} u^{(k-m)j} = \delta_{km}$. Check of the other claims: We have

$$\begin{aligned} 2) \quad e_k e_m &= \sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{kj} K^j e_m = \left(\sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{(k-m)j} \right) e_m = \delta_{km} e_k \\ 3) \quad K e_k &= \frac{1}{2N} K \left(\sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{kj} K^j \right) \\ &= \frac{u^{-k}}{2N} \sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{k(j+1)} K^{j+1} = u^{-k} e_k \\ 5) \quad E e_k &= \frac{1}{2N} E \left(\sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{kj} K^j \right) = \frac{1}{2N} \sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{(N+k)j} K^j E = e_{k+N} E \\ 6) \quad e_k F &= \frac{1}{2N} \left(\sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{kj} K^j \right) F = \frac{1}{2N} F \sum_{j \in \mathbb{Z}/2N\mathbb{Z}} u^{(N+k)j} K^j = F e_{k+N} \end{aligned}$$

Let us now describe in detail the two cases $N = 1$ and $N = 2$.

2.1 Case $N = 1$

For $N = 1$ we have $K^2 = \mathbf{1}$, hence $K = K^{-1}$.

Definition 2.2 *The algebra* H_1^i *is defined by the symbols* K, E, F *and the relations*

$$\left\{ \begin{array}{c} KE = -EK, \\ KF = -FK, \\ [E, F] = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} E^2 = 0, \\ F^2 = 0, \\ K^2 = \mathbf{1}. \end{array} \right. \tag{12}$$

Lemma 2.3 *1) With*

$$\begin{cases} e_0 = \frac{\mathbf{1}+\mathbf{K}}{2}, \\ e_1 = \frac{\mathbf{1}-\mathbf{K}}{2}, \end{cases} \tag{13}$$

H_1^i is equivalently defined by the symbols e_0, e_1, E, F, K and the relations

$$\begin{cases} e_0^2 = e_0, \\ e_1^2 = e_1, \\ e_1 e_0 = 0, \\ e_0 + e_1 = \mathbf{1}, \end{cases} \quad \begin{cases} E^2 = 0, \\ F^2 = 0, \\ [E,F] = 0, \end{cases} \quad \begin{cases} e_0 E = E e_1, \\ e_1 E = E e_0, \\ e_0 F = F e_1, \\ e_1 F = F e_0, \end{cases} \tag{14}$$

yielding the basis

$$\begin{cases} e_0, \\ e_1, \\ E_0 = e_0 E, \\ E_1 = e_1 E, \\ F_0 = e_0 F, \\ F_1 = e_1 F, \\ C_0 = e_0 EF, \\ C_1 = e_1 EF. \end{cases} \tag{15}$$

2) H_1^i is a $*$-algebra under the $*$-operation specified by

$$\begin{cases} E^* = F, \\ F^* = E, \\ K^* = K, \end{cases} \qquad \begin{cases} e_0^* = e_0, \\ e_1^* = e_1. \end{cases}$$

3) The Casimir element is

$$C = C^* = EF = FE.$$

4) The basis 15 is acted upon as follows by multiplications by E, F and C:

$$\begin{cases} Ee_0 = E_1, \\ Ee_1 = E_0, \\ EE_0 = 0, \\ EE_1 = 0, \\ EF_0 = C_1, \\ EF_1 = C_0, \\ EC_0 = 0, \\ EC_1 = 0, \end{cases} \begin{cases} Fe_0 = F_1, \\ Fe_1 = F_0, \\ FE_0 = C_1, \\ FE_1 = C_0, \\ FF_0 = 0, \\ FF_1 = 0, \\ FC_0 = 0, \\ FC_1 = 0, \end{cases} \begin{cases} e_0 E = E_0, \\ e_1 E = E_1, \\ E_0 E = 0, \\ E_1 E = 0, \\ F_0 E = C_0, \\ F_1 E = C_1, \\ C_0 E = 0, \\ C_1 E = 0, \end{cases} \begin{cases} e_0 F = F_0, \\ e_1 F = F_1, \\ E_0 F = C_0, \\ E_1 F = C_1, \\ F_0 F = 0, \\ F_1 F = 0, \\ C_0 F = 0, \\ C_1 F = 0, \end{cases} \begin{cases} Ce_0 = C_0, \\ Ce_1 = C_1, \\ CE_0 = 0, \\ CE_1 = 0, \\ CF_0 = 0, \\ CF_1 = 0, \\ CC_0 = 0, \\ CC_1 = 0. \end{cases} \tag{16}$$

5) The multiplication table of H_1^i is as follows (we plugged the product XY at the intersection of line X and column Y, the latter so as to have $*$-symmetry w.r.t. the diagonal).

	e_0	e_1	E_0	E_1	F_0	F_1	C_0	C_1
e_0	e_0	0	E_0	0	F_0	0	C_0	0
e_1	0	e_1	0	E_1	0	F_1	0	C_1
E_0	0	E_0	0	0	0	C_0	0	0
E_1	E_1	0	0	0	C_1	0	0	0
F_0	0	F_0	0	C_0	0	0	0	0
F_1	F_1	0	C_1	0	0	0	0	0
C_0	C_0	0	0	0	0	0	0	0
C_1	0	C_1	0	0	0	0	0	0

(17)

Proof:

Immediate from relations 12 and 14. □

2.2 Structure of the algebra H_1^i

Proposition 2.2 *1) Let* $\mathbf{M_2}$ *be algebra of the* 2×2 *complex matrices, equipped with its natural grading (diagonal entries are even and off-diagonal entries odd) and* $\Lambda = \Lambda_1 \otimes \Lambda_1$ *with* Λ_1 *the Grassmann algebra over* $\mathbb{C}$ *(ordinary-not skew-tensor product equipped with the tensor product* $\mathbb{Z}/2\mathbb{Z}$*-grading).*

As an algebra H_1^i *is isomorphic to the even part* $(\mathbf{M_2} \otimes \mathbf{\Lambda})^+$ *(again ordinary-not skew- tensor .product equipped with the tensor product* $\mathbb{Z}/2\mathbb{Z}$*-grading). This isomorphism is specified as follows. With* $\mathbf{M_2}$ *spanned by its matrix units* $\{e_{lk}\}_{l,k=0,1}$*; the first* Λ_1*-factor by* $\mathbf{1}, \mathbf{e}$*; the second factor by* $\mathbf{1}, \mathbf{f}$*; and the tensor product* $\Lambda_1 \otimes \Lambda_1$ *by* $\mathbf{1}\otimes\mathbf{1}, \mathbf{E} = \mathbf{e}\otimes\mathbf{1}, \mathbf{F} = \mathbf{1}\otimes\mathbf{f}, \mathbf{EF} = \mathbf{e}\otimes\mathbf{f}$*, one has,*

$$\begin{cases} e_0 = e_{00} \otimes \mathbf{1}, & E_0 = e_{01} \otimes E, \quad ,F_0 = e_{00} \otimes F, \quad ,C_0 = e_{00} \otimes EF, \\ e_1 = e_{11} \otimes \mathbf{1}, & E_1 = e_{10} \otimes E, \quad F_1 = e_{11} \otimes F, \quad C_1 = e_{11} \otimes EF. \end{cases}$$

2) The subspace N_i^1 *of* H_1^i *spanned by* E_0, E_1, F_0, F_1, C_0 *and* C_1 *is the latter's nilradical, giving rise to the quotient* $H_1^i/N_i^1 \cong \mathbb{C} \oplus \mathbb{C}$*.*

3) We definine as follows the scalar product $< .,. >$*,*

$$< a, b >= Tr\lambda(a^* b) \qquad a, b \in H_1^i$$

where λ *denotes the left-regular representation of* H_1^i*.* $< .,. >$ *is positive semi-definite with null-space the nilradical* N_i^1*. It is positive definite on the span of* e_0, e_1*, where it coincides with the usual trace of* $\mathbb{C} \oplus \mathbb{C}$ *(*$< e_l, e_k >= \delta_{lk},\ l, k = 0, 1$*).*

Proof:

1) One immediately checks that the elements 15 fulfill the multiplication rules 16.

2) These multiplication rules imply that N_i^1 is a subalgebra fulfilling the inclusions $e_0 N_1^i, N_1^i e_0, e_1 N_1^i, N_1^i e_1 \subset N_1^i$, N_1^i is thus an ideal of H_1^i which moreover consists of nilpotent elements and yields a quotient generated by e_0, e_1 fulfilling $e_0^2 = e_0$, $e_1^2 = e_1$, $e_0 e_1 = 0$, thus isomorphic to the semi-simple $\mathbb{C} \oplus \mathbb{C}$. Accordingly, N_i^1 is the nilradical of H_1^i.

3) Clear by inspection of 16, implying first that $Tr(\rho)$ vanishes on N_i^1 and that $Tr\rho(e_0) = Tr\rho(e_1) = 1$. □

2.3 Case $N = 2$

We now describe the case corresponding to $N = 2$, one has now $K^4 = \mathbf{1}$ and $K - K^{-1}$ no longer vanishes.

Definition 2.3 *The algebra* H_2^i *is defined by the symbols* K, E *and* F *together with the relations*

$$\begin{cases} KE = -EK, \\ KF = -FK, \\ [E, F] = \frac{1}{2i}(K - K^{-1}), \end{cases} \qquad \begin{cases} E^2 = 0, \\ F^2 = 0, \\ K^4 = \mathbf{1}. \end{cases}$$

Lemma 2.4 *1) Let* $e_m = \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} i^{mk} K^k$*, i.e*

$$\begin{cases} e_0 = \frac{1}{4}\left(\mathbf{1} + \mathbf{K} + \mathbf{K^2} + \mathbf{K^3}\right), \\ e_1 = \frac{1}{4}\left(\mathbf{1} + \mathbf{iK} - \mathbf{K^2} - \mathbf{iK^3}\right), \\ e_2 = \frac{1}{4}\left(\mathbf{1} - \mathbf{K} + \mathbf{K^2} - \mathbf{K^3}\right), \\ e_3 = \frac{1}{4}\left(\mathbf{1} - \mathbf{K} - \mathbf{K^2} + \mathbf{iK^3}\right). \end{cases}$$

It implies the relations

$$\begin{cases} K = e_0 - ie_1 - e_2 + ie_3, \\ K^{-1} = e_0 + ie_1 - e_2 - ie_3, \\ K - K^{-1} = 2i(e_3 - e_1), \end{cases}$$

yielding the basis

$$\left\{\begin{array}{c} e_0, \\ e_2, \\ E_0 = e_0 E, \\ E_2 = e_2 E, \\ F_0 = e_0 F, \\ F_2 = e_2 F, \\ P_0 = C_0 = e_0 EF, \\ P_2 = C_2 = e_2 EF, \end{array}\right. \qquad \left\{\begin{array}{c} e_1, \\ e_3, \\ E_1 = e_1 E, \\ E_3 = e_3 E, \\ F_1 = e_1 F, \\ F_3 = e_3 F, \\ P_1 = C_1 - \frac{1}{2} e_1 = e_1 EF, \\ P_3 = C_3 + \frac{1}{2} e_3 = e_3 EF. \end{array}\right. \tag{18}$$

Accordingly, H_2^i *is equivalently defined by the symbols* E*,* F *and* e_j $j = 0,1,2,3$ *and the relations*

$$\left\{\begin{array}{c} e_0 + e_1 + e_2 + e_3 = \mathbf{1}, \\ e_l e_k = \delta_{lk}, \end{array}\right. , l = 0,1,2,3 \left\{\begin{array}{c} E^2 = 0, \\ F^2 = 0, \\ [E, F] = e_3 - e_1, \end{array}\right. \quad \left\{\begin{array}{c} e_j E = E e_{j+2}, \\ F e_j = e_{j+2} F, \end{array}\right. \ j \in \mathbb{Z}/4\mathbb{Z}.$$

2) H_2^i *is a* $*$*-algebra under the* $*$*-operation specified by*

$$\left\{\begin{array}{c} E^* = F, \\ F^* = E, \end{array}\right. \qquad \left\{\begin{array}{c} K^* = K, \\ e_j^* = e_j, \end{array}\right. \ j = 0,1,2,3.$$

3) The Casimir element is

$$C = C^* = FE - \frac{1}{2}(e_1 - e_3) = EF + \frac{1}{2}(e_1 - e_3). \tag{19}$$

4) Let

$$\left\{\begin{array}{c} \pi_0 = e_0 + e_2, \\ \pi_1 = e_1 + e_3, \end{array}\right. , \left\{\begin{array}{c} H_2^{i(0)} = \pi_0 H_2^i, \\ H_2^{i(1)} = \pi_1 H_2^i. \end{array}\right.$$

Then π_0 *and* π_1 *are supplementary central idempotents yielding supplementary ideals* $H_2^{i(0)}$ *and* $H_2^{i(1)}$ *of* H_2^i *spanned respectively by the left and right basis* 18.

5) Multiplication by E*,* F *and* C *act as follows on the basis* 18*,*

$$\left\{\begin{array}{c} Ee_0 = E_2, \\ Ee_2 = E_0, \\ EE_0 = 0, \\ EE_2 = 0, \\ EF_0 = P_2, \\ EF_2 = P_0, \\ EP_0 = 0, \\ EP_1 = 0, \end{array}\right. \left\{\begin{array}{c} Fe_0 = F_2, \\ Fe_2 = F_0, \\ FE_0 = P_2, \\ FE_2 = P_0, \\ FF_0 = 0, \\ FF_2 = 0, \\ FP_0 = 0, \\ FP_1 = 0. \end{array}\right. \left\{\begin{array}{c} e_0 E = E_0, \\ e_2 E = E_2, \\ E_0 E = 0, \\ E_2 E = 0, \\ F_0 E = P_0, \\ F_2 E = P_2, \\ P_0 E = 0, \\ P_2 E = 0, \end{array}\right. \left\{\begin{array}{c} e_0 F = F_0, \\ e_2 F = F_2, \\ E_0 F = P_0, \\ E_2 F = P_2, \\ F_0 F = 0, \\ F_2 F = 0, \\ P_0 F = 0, \\ P_1 F = 0, \end{array}\right. \left\{\begin{array}{c} Pe_0 = P_0, \\ Pe_2 = P_2, \\ PE_0 = 0, \\ PE_2 = 0, \\ PF_0 = 0, \\ PF_2 = 0, \\ PP_0 = 0, \\ PP_2 = 0, \end{array}\right.$$

and

$$\left\{\begin{array}{c} Ee_1 = E_3, \\ Ee_3 = E_1, \\ EE_1 = 0, \\ EE_3 = 0, \\ EF_1 = P_3, \\ EF_3 = P_1, \\ EP_1 = 0, \\ EP_3 = 0, \end{array}\right. \left\{\begin{array}{c} Fe_1 = F_3, \\ Fe_3 = F_1, \\ FE_1 = P_3, \\ FE_3 = P_1, \\ FF_1 = 0, \\ FF_3 = 0, \\ FP_1 = 0, \\ FP_3 = 0, \end{array}\right. \left\{\begin{array}{c} e_1 E = E_1, \\ e_3 E = E_3, \\ E_1 E = 0, \\ E_3 E = 0, \\ F_1 E = P_1 + e_1, \\ F_3 E = P_3 - e_3, \\ P_1 E = -E_1, \\ P_3 E = E_3, \end{array}\right. \left\{\begin{array}{c} e_1 F = F_1, \\ e_3 F = F_3, \\ E_1 F = P_1, \\ E_3 F = P_3, \\ F_1 F = 0, \\ F_3 F = 0, \\ P_1 F = 0, \\ P_3 F = 0, \end{array}\right. \left\{\begin{array}{c} Ce_1 = C_1, \\ Ce_3 = C_3, \\ CE_1 = -\frac{1}{2} E_1, \\ CE_3 = \frac{1}{2} E_3, \\ CF_1 = \frac{1}{2} F_1, \\ CF_3 = -\frac{1}{2} F_3, \\ CP_1 = -\frac{1}{2} P_1, \\ CP_3 = \frac{1}{2} P_3. \end{array}\right.$$

6) One has the following multiplication tables (the product XY *is at the intersection of line* X *and column* Y*, ordering the latter so as to have* $*$ *symmetry w.r.t the diagonal) for* $H_2^{i(0)}$ *in the basis* $\{e_0, e_2, E_0, E_2, F_0, F_2, P_0, P_2\}$

	e_0	e_2	E_0	E_2	F_0	F_2	P_0	P_2
e_0	e_0	0	E_0	0	F_0	0	P_0	0
e_2	0	e_2	0	E_2	0	F_2	0	P_2
E_0	0	E_0	0	0	0	P_0	0	0
E_2	E_2	0	0	0	P_2	0	0	0
F_0	0	F_0	0	P_0	0	0	0	0
F_2	F_2	0	P_2	0	0	0	0	0
P_0	P_0	0	0	0	0	0	0	0
P_2	0	P_2	0	0	0	0	0	0

(20)

and for $H_2^{i(1)}$ *in the basis* $\{e_1, e_3, E_1, E_3, F_1, F_3, P_1, P_3\}$

	e_1	e_3	E_1	E_3	F_1	F_3	P_1	P_3
e_1	e_1	0	E_1	0	F_1	0	P_1	0
e_3	0	e_3	0	E_3	0	F_3	0	P_3
E_1	0	E_1	0	0	0	P_1	0	0
E_3	E_3	0	0	0	P_3	0	0	0
F_1	0	F_1	0	$P_1 + e_1$	0	0	0	F_1
F_3	F_3	0	$P_3 - e_3$	0	0	0	$-F_3$	0
P_1	P_1	0	$-E_1$	0	0	0	$-P_1$	0
P_3	0	P_3	0	E_3	0	0	0	P_3

(21)

This yields the following action of the Casimir operator

x	e_1	e_3	E_1	E_3	F_1	F_3	P_1	P_3
$C(x)$	C_1	C_3	$-\frac{1}{2}E_1$	$\frac{1}{2}E_3$	$\frac{1}{2}F_1$	$-\frac{1}{2}E_3$	$-\frac{1}{2}P_1$	$\frac{1}{2}P_3$

and shows that the restriction of C *to* $H_2^{i(1)}$ *has the eigenspaces* $\mathbb{C}E_1 + \mathbb{C}F_3 + \mathbb{C}P_1$ *to the eigenvalue* $-\frac{1}{2}$ *and* $\mathbb{C}E_3 + \mathbb{C}F_1 + \mathbb{C}P_3$ *to the eigenvalue* $\frac{1}{2}$.

Proof:

The products 20 and 21 not involving the C_j are immediate from 12 and 14. Check of the products 21 involving P_1, P_3 is made using

$$\begin{cases} EFE = -(e_1 - e_3)E, \\ FEF = (e_1 - e_3)F, \end{cases}$$

so that,

$$\begin{cases} E_1F_3 = e_1EF = P_1, \\ F_3E_1 = e_3FE = e_3(EF + e_1 - e_3) = P_3 - e_3, \end{cases}$$

$$\begin{cases} F_3P_1 = e_3FEF = e_3(e_1 - e_3)F = -F_3, \\ P_1E_1 = e_1EFE = -e_1(e_1 - e_3)E = -E_1, \\ P_1P_1 = e_1EFEF = -e_1(e_1 - e_3)EF = -e_1EF, \end{cases}$$

$$\begin{cases} E_3F_1 = e_3EF = P_3, \\ F_1E_3 = e_1FE = e_1(EF + e_1 - e_3) = P_1 + e_1, \\ F_1P_3 = e_1FEF = e_1(e_1 - e_3)F = e_1F = F_1, \end{cases}$$

$$\begin{cases} P_3E_1 = e_3EFE = -e_3(e_1 - e_3)E = e_3E = E_3, \\ P_3E_3 = e_3EFE = -e_3(e_1 - e_3)E = e_3E = E_3, \\ P_3P_3 = e_3EFEF = -e_3(e_1 - e_3)EF = e_3EF = P_3. \end{cases}$$

□

2.4 Structure of the algebra H_2^i

Proposition 2.3 *1) The algebra H_2^i splits into the direct sum of the ideals* [1] $H_2^{i(0)}$ *and* $H_2^{i(1)}$.

2) The algebra $H_2^{i(0)}$ is isomorphic to H_1^i with the isomorphism given by

$$\left\{\begin{array}{l} e_0 \to e_0, \\ e_2 \to e_1, \end{array}\right. \left\{\begin{array}{l} E_0 \to E_0, \\ E_2 \to E_1, \end{array}\right. \left\{\begin{array}{l} F_0 \to F_0, \\ F_2 \to F_1, \end{array}\right. \left\{\begin{array}{l} C_0 \to C_0, \\ C_2 \to C_1. \end{array}\right. \tag{22}$$

Consequently, $H_2^{i(0)}$ is an algebra isomorphic to the even part $(\mathbf{M_2} \otimes \mathbf{\Lambda_1} \otimes \mathbf{\Lambda_1})^+$, the isomorphism being specified as follows : with $\mathbf{M_2}$ spanned by its matrix units $\{e_{lk}\}_{l,k=0,1}$; the first Λ_1-factor by $\mathbf{1}$, e, the second Λ_1-factor by $\mathbf{1}$, f; and the tensor product $\Lambda_1 \otimes \Lambda_1$ by $\mathbf{1} \otimes \mathbf{1}, \mathbf{E} = \mathbf{e} \otimes \mathbf{1}, \mathbf{F} = \mathbf{1} \otimes \mathbf{f}, \mathbf{EF} = \mathbf{e} \otimes \mathbf{f}$, one has

$$\left\{\begin{array}{llll} e_1 = e_{00} \otimes \mathbf{1}, & E_0 = e_{01} \otimes E, & F_0 = e_{01} \otimes F, & P_0 = e_{00} \otimes EF, \\ e_2 = e_{11} \otimes \mathbf{1}, & E_2 = e_{10} \otimes E, & F_2 = e_{10} \otimes F, & P_2 = e_{11} \otimes EF. \end{array}\right.$$

3) The nilradical $N_1^{i(0)}$ of $H_2^{i(0)}$ is the eigenspace of the Casimir element C to the eigenvalue 0, it is spanned by the elements $E_0, E_2, F_0, F_2, P_0, P_2$. The quotient algebra $H_2^{i(0)}/N_1^{i(0)}$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$.

4)$H_2^{i(1)}$ is an algebra isomorphic to the semi-simple algebra $M(2,\mathbb{C}) \oplus M(2,\mathbb{C})$ with the isomorphism given by

$$K = i\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5)We define the scalar product $< .,. >$,

$$< a, b >= Tr\lambda(a^* b),$$

where λ denotes the left regular representation of H_2^i. One has that

a) the even and odd parts $H_2^{i(0)}$ and $H_2^{i(1)}$ are mutually $< .,. >$-orthogonal,

b) its restriction to $H_2^{i(0)}$, $< .,. >$ is positive semi-definite with null space $N_1^{i(0)}$, and is positive definite on the span of e_0, e_2, where it coincides with the usual trace of $\mathbb{C} \oplus \mathbb{C}$,

c) its restriction to $H_2^{i(1)}$, $< .,. >$ behaves as follows with

$$\begin{aligned} H_2^{i(1,+)} &= \text{span of } \left\{\frac{1}{2}(e_1 - P_1), \frac{1}{2}(e_3 + P_3), F_1\right\}, \\ H_2^{i(1,0)} &= \text{span of } \left\{\frac{1}{2}(e_1 + P_1), \frac{1}{2}(e_3 - P_3), E_1, E_3\right\}, \\ H_2^{i(1,-)} &= \text{span of } \{F_3\}, \end{aligned}$$

$< .,. >$ is positive definite on $H_2^{i(1,+)}$, with $\frac{1}{2}(e_1 - P_1), \frac{1}{2}(e_3 + P_3), F_1$ orthonormal,
$< .,. >$ has $H_2^{i(1,0)}$ as its null space (space of vectors orthogonal to all vectors),
$< .,. >$ is negative definite on $H_2^{i(1,-)}$, with $< F_3, F_3 >= -1$.

Proof:

1) Recalls a former result, cf lemma 11.2.

2) The changes 22 turn 20 into 17.

3) The eigenspaces of C acting on $H_2^{i(0)}$ are immediately found computing 19 via 20. The (a priori known) fact that these subspaces are ideals and the corresponding quotient are patent from 20 and 21.

4)Follows from the fact that these matrices satisfy the same relations and generate an 8-dimensional algebra.

5) Inspection of 21 yields the following table of values of $Tr\lambda(.)$ on $H_2^{i(1)}$

[1] algebraic (not Hopf) ideals.

u	e_1	e_3	E_1	E_3	F_1	F_3	P_1	P_3
$Tr\lambda(u)$	1	1	0	0	1	0	0	-1

Thus we have the following table of values of the scalar product $< .,. >$ on $H_2^{i(1)}$ (we plotted $< u, v >$ at the intersection of line u and v

	e_1	e_3	E_1	E_3	F_1	F_3	P_1	P_3
e_1	1	0	0	0	0	0	-1	0
e_3	0	1	0	0	0	0	0	1
E_1	0	0	0	0	0	0	0	0
E_3	0	0	0	0	0	0	0	0
F_1	0	0	0	0	1	0	0	0
F_3	0	0	0	0	0	-1	0	0
P_1	-1	0	0	0	0	0	1	0
P_3	0	1	0	0	0	0	0	1

The statement then immediately follows. $\square$

3 Hopf structure of H_1^i and H_2^i

Note that 2-4 give the Hopf structure of H_1^i and H_2^i. Recalling basic definitions, the algebra H_1^i is defined by symbols K, E, F and the relations

$$\begin{cases} KE = -EK, \\ KF = -FK, \\ [E, F] = 0, \end{cases} \begin{cases} E^2 = 0, \\ F^2 = 0, \\ K^2 = \mathbf{1}, \end{cases} \begin{cases} e_l e_k = \delta_{lk}, \quad l, k = 0, 1 \\ e_0 + e_1 = \mathbf{1}, \end{cases} \begin{cases} e_j E = E e_{j+1}, \quad j \in \mathbb{Z}/2\mathbb{Z} \\ e_j F = F e_{j+1}. \quad j \in \mathbb{Z}/2\mathbb{Z} \end{cases}$$

It is spanned by the basis

$$e_0 = \frac{\mathbf{1+K}}{2}, e_1 = \frac{\mathbf{1-K}}{2}, E_0 = e_0 E, E_1 = e_1 E, F_0 = e_0 F, C_0 = e_0 EF, C_1 = e_1 EF.$$

3.1 Hopf structure of H_1^i

Proposition 3.1 *The coproduct is given by*

$$\begin{cases} \Delta(e_0) = e_0 \otimes e_0 + e_1 \otimes e_1, \\ \Delta(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0, \\ \Delta(E_0) = E_0 \otimes e_0 + E_1 \otimes e_1 + e_0 \otimes E_0 - e_1 \otimes E_1, \\ \Delta(E_1) = E_0 \otimes e_1 + E_1 \otimes e_0 + e_0 \otimes E_1 - e_1 \otimes E_0, \\ \Delta(F_0) = F_0 \otimes e_0 - F_1 \otimes e_1 + e_0 \otimes F_0 + e_1 \otimes F_1, \\ \Delta(F_1) = -F_0 \otimes e_1 + F_1 \otimes e_0 + e_0 \otimes F_1 + e_1 \otimes F_0, \\ \Delta(C_0) = C_0 \otimes e_0 - C_1 \otimes e_1 + e_0 \otimes C_0 - e_1 \otimes C_1 + E_0 \otimes F_0 + E_1 \otimes F_1 - F_0 \otimes E_0 - F_1 \otimes E_1, \\ \Delta(C_1) = -C_0 \otimes e_1 + C_1 \otimes e_0 + e_0 \otimes C_1 - e_1 \otimes C_0 + F_0 \otimes E_1 + F_1 \otimes E_0 + E_0 \otimes F_1 + E_1 \otimes F_0. \end{cases} \tag{23}$$

The antipde and the counit are

$$\begin{cases} S(e_0) = e_0, \\ S(e_1) = e_1, \\ S(E_0) = E_1, \\ S(E_1) = -E_0, \end{cases} \begin{cases} S(F_0) = -F_1, \\ S(F_1) = F_0, \\ S(C_0) = C_0, \\ S(C_1) = C_1, \end{cases} \tag{24}$$

$$\begin{cases} \varepsilon(e_0) = 1, \\ \varepsilon(e_1) = \varepsilon(E_0) = \varepsilon(E_1) = \varepsilon(F_0) = \varepsilon(F_1) = \varepsilon(C_0) = \varepsilon(C_1) = 0. \end{cases} \tag{25}$$

Proof:

Check of 23 : Using 13, whence $K = e_0 - e_1$, we have

$$\begin{aligned}\Delta(e_0) &= \frac{1}{2}\Delta(\mathbf{1}+\mathbf{K}) = \frac{\mathbf{1}}{\mathbf{2}}[\mathbf{1}\otimes\mathbf{1}+\mathbf{K}\otimes\mathbf{K}] = \frac{\mathbf{1}}{\mathbf{2}}[(\mathbf{e_0}+\mathbf{e_1})\otimes(\mathbf{e_0}+\mathbf{e_1})+(\mathbf{e_0}-\mathbf{e_1})\otimes(\mathbf{e_0}-\mathbf{e_1})] \\ &= e_0\otimes e_0 + e_1\otimes e_1,\end{aligned}$$

$$\begin{aligned}\Delta(e_1) &= \frac{1}{2}\Delta(\mathbf{1}-\mathbf{K}) = \frac{\mathbf{1}}{\mathbf{2}}[\mathbf{1}\otimes\mathbf{1}-\mathbf{K}\otimes\mathbf{K}] = \frac{\mathbf{1}}{\mathbf{2}}[(\mathbf{e_0}+\mathbf{e_1})\otimes(\mathbf{e_0}+\mathbf{e_1})-(\mathbf{e_0}-\mathbf{e_1})\otimes(\mathbf{e_0}-\mathbf{e_1})] \\ &= e_0\otimes e_1 + e_1\otimes e_0\end{aligned}$$

$$\begin{aligned}\Delta(E_0) &= \Delta(e_0E) = \Delta(e_0)\Delta(E) = (e_0\otimes e_0 + e_1\otimes e_1)\,(E\otimes(e_0+e_1)+(e_0-e_1)\otimes E) \\ &= E_0\otimes e_0 + E_1\otimes e_1 + e_0\otimes E_0 - e_1\otimes E_1,\end{aligned}$$

$$\begin{aligned}\Delta(E_1) &= \Delta(e_1E) = \Delta(e_1)\Delta(E) = (e_0\otimes e_1 + e_1\otimes e_0)\,(E\otimes(e_0+e_1)+(e_0-e_1)\otimes E) \\ &= E_0\otimes e_1 + E_1\otimes e_0 + e_0\otimes E_1 - e_1\otimes E_0.\end{aligned}$$

Further, taking account of the fact that $K^{-1} = K = e_0 - e_1$

$$\begin{aligned}\Delta(F_0) &= \Delta(e_0F) = \Delta(e_0)\Delta(F) = (e_0\otimes e_0 + e_1\otimes e_1)\,(F\otimes(e_0-e_1)+(e_0+e_1)\otimes F) \\ &= F_0\otimes e_0 - F_1\otimes e_1 + e_0\otimes F_0 + e_1\otimes F_1,\end{aligned}$$

$$\begin{aligned}\Delta(F_1) &= \Delta(e_1F) = \Delta(e_1)\Delta(F) = (e_0\otimes e_1 + e_1\otimes e_0)\,(F\otimes(e_0-e_1)+(e_0+e_1)\otimes F) \\ &= -F_0\otimes e_1 + F_1\otimes e_0 + e_0\otimes F_1 + e_1\otimes F_0,\end{aligned}$$

$$\begin{aligned}\Delta(C_0) &= \Delta(E_0)\Delta(F) \\ &= (E_0\otimes e_0 + E_1\otimes e_1 + e_0\otimes E_0 - e_1\otimes E_1)\,(F\otimes(e_0-e_1)+(e_0+e_1)\otimes F) \\ &= C_0\otimes e_0 - C_1\otimes e_1 + e_0\otimes C_0 - e_1\otimes C_1 + E_0\otimes F_0 + E_1\otimes F_1 - F_0\otimes E_0 - F_1\otimes E_1,\end{aligned}$$

$$\begin{aligned}\Delta(C_1) &= \Delta(E_1)\Delta(F) \\ &= (E_0\otimes e_1 + E_1\otimes e_0 + e_0\otimes E_1 - e_1\otimes E_0)\,(F\otimes(e_0-e_1)+(e_0+e_1)\otimes F) \\ &= -C_0\otimes e_1 + C_1\otimes e_0 + e_0\otimes C_1 - e_1\otimes C_0 + F_0\otimes E_1 + F_1\otimes E_0 + E_0\otimes F_1 + E_1\otimes F_0.\end{aligned}$$

Check of 24 : Using again $K = e_0 - e_1$, we have

$$\begin{aligned}S(e_0) &= \frac{1}{2}S(\mathbf{1}+\mathbf{K}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\mathbf{K^{-1}}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\mathbf{K^{-1}}) = \mathbf{e_0} \\ S(e_1) &= \frac{1}{2}S(\mathbf{1}-\mathbf{K}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}-\mathbf{K^{-1}}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}-\mathbf{K^{-1}}) = \mathbf{e_1}.\end{aligned}$$

Since S is antiisomorphism, we also have

$$\begin{aligned}
S(E_0) &= S(e_0E) = S(E)S(e_0) = (-K^{-1}E)e_0 = -(e_0-e_1)\,Ee_0 = E_1,\\
S(E_1) &= S(e_1E) = S(E)S(e_1) = (-K^{-1}E)e_1 = -(e_0-e_1)\,Ee_1 = -E_0,\\
S(F_0) &= S(e_0F) = S(F)S(e_0) = (-FK)e_0 = -F\,(e_0-e_1)\,e_0 = -F_1,\\
S(F_1) &= S(e_1F) = S(F)S(e_1) = (-FK)e_1 = -F\,(e_0-e_1)\,e_1 = F_0,\\
S(C_0) &= S(e_0EF) = S(F)S(E)S(e_0) = F\,(e_0-e_1)\,(e_0-e_1)\,Ee_0 = Fe_1E = e_0EF = C_0,\\
S(C_1) &= S(e_1EF) = S(F)S(E)S(e_1) = F\,(e_0-e_1)\,(e_0-e_1)\,Ee_1 = Fe_0E = e_1EF = C_1.
\end{aligned}$$

Check of 25 : using again $K = e_0 - e_1$, we have

$$\begin{aligned}
\varepsilon(e_0) &= \frac{1}{2}\varepsilon(\mathbf{1}+\mathbf{K}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\mathbf{1}) = \mathbf{1},\\
\varepsilon(e_1) &= \frac{1}{2}\varepsilon(\mathbf{1}-\mathbf{K}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}-\mathbf{1}) = \mathbf{0}.
\end{aligned}$$

Since ε is a morphism, we get, typically:,

$$\varepsilon(E_0) = \varepsilon(e_0)\varepsilon(E) = 1\times 0 = 0.$$

□

Remark 3.1 *The eigenvalues of S in H_1^i are 1, i and $-i$ with the eigenspaces*

$$V_1 \text{ spanned by } \left\{\begin{array}{c} e_0\\ e_1\\ C_0\\ C_1\end{array}\right.,\ V_i \text{ spanned by } \left\{\begin{array}{c} E_0-iE_1\\ F_0+iF_1\end{array}\right. \text{ and } V_{-i} \text{ spanned by } \left\{\begin{array}{c} E_0+iE_1\\ F_0-iF_1\end{array}\right..$$

3.2 Hopf structure of H_2^i

Recall that the algebra H_2^i is defined by the symbols K, E, F and the relations

$$\left\{\begin{array}{c} KE = -EK,\\ KF = -FK,\\ [E,F] = \frac{1}{2i}(K-K^{-1}),\end{array}\right. \qquad \left\{\begin{array}{l} E^2 = 0,\\ F^2 = 0,\\ K^4 = \mathbf{1},\end{array}\right.$$

or else symbols E, F and $e_m = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/4\mathbb{Z}} i^{mk}K^k$, $i = 0,1,2,3$, and relations

$$\left\{\begin{array}{c} \sum\limits_{m=0}^{3} e_m = \mathbf{1},\\ e_le_m = \delta_{lm},\quad l,m = 0,1,2,3\end{array}\right. \quad \left\{\begin{array}{c} E^2 = 0,\\ F^2 = 0,\\ [E,F] = e_3 - e_1,\end{array}\right. \quad \left\{\begin{array}{l} e_jE = Ee_{j+2},\\ Fe_j = e_{j+2}F,\end{array}\right. \quad j = 0,1,2,3.$$

It is spanned by $e_0, e_2, E_0, E_2, F_0, F_2, P_0, P_2, e_1, e_3, E_1, E_3, F_1, F_3, P_1, P_3$.

Proposition 3.2 *We have, for $m \in \mathbb{Z}/4\mathbb{Z}$,*

$$\left\{\begin{array}{l}
\Delta(e_m) = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/4\mathbb{Z}} e_k\otimes e_{m-k},\\
\Delta(E_m) = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/4\mathbb{Z}} \left[E_k\otimes e_{m-k} + (i)^{-k}e_k\otimes E_{m-k}\right],\\
\Delta(F_m) = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/4\mathbb{Z}} \left[(i)^{m-k}F_k\otimes e_{m-k} + e_k\otimes F_{m-k}\right],\\
\Delta(P_m) = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/4\mathbb{Z}} \left[i^{m-k}P_k\otimes e_{m-k} + E_k\otimes F_{m-k} - (-1)^k(i)^m F_k\otimes E_{m-k} + (i)^{-k}e_k\otimes P_{m-k}\right],
\end{array}\right. \tag{26}$$

$$\left\{\begin{array}{l} S(e_m) = e_{-m}, \\ S(E_m) = (-i)^m E_{2-m}, \\ S(F_m) = -(i)^m F_{2-m}, \\ S(P_m) = P_{-m} - \delta_{3m} e_1 + \delta_{1m} e_3, \end{array}\right. \tag{27}$$

$$\left\{\begin{array}{l} \varepsilon(e_m) = \delta_{0m} \\ \varepsilon(E_m) = \varepsilon(F_m) = \varepsilon(P_m) = 0 \end{array}\right. \tag{28}$$

Proof:

Check of 26: Using $K = e_0 - ie_1 - e_2 + ie_3$, $\sum\limits_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^{kl} = 4\delta_{0l}$ for $l \in \mathbb{Z}/4\mathbb{Z}$, $Ke_k = e_k K = (i)^{-k} e_k$, $K^{-1}e_k = e_k = K^{-1}(i)^k e_k$, we have

$$\begin{aligned} \Delta(e_m) &= \frac{1}{4} \sum_{r \in \mathbb{Z}/4\mathbb{Z}} (i)^{mr} \Delta(K)^r = \frac{1}{4} \sum_{l \in \mathbb{Z}/4\mathbb{Z}} (i)^{mr} K^r \otimes K^r = \frac{1}{4} \sum_{r,k,l \in \mathbb{Z}/4\mathbb{Z}} (i)^{mr} \left((i)^{-kr} e_k \otimes (i)^{-lr} e_l\right) \\ &= \frac{1}{4} \sum_{r,k,l \in \mathbb{Z}/4\mathbb{Z}} (i)^{r(m-k-l)} \left(e_k \otimes e_l\right) = \frac{1}{4} \sum_{k,l \in \mathbb{Z}/4\mathbb{Z}} \delta_{m,k+l} \left(e_k \otimes e_l\right) = \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left(e_k \otimes e_{m-k}\right), \end{aligned}$$

$$\begin{aligned} \Delta(E_m) &= \Delta(e_m)\Delta(E) = \frac{1}{4} \left(\sum_{k \in \mathbb{Z}/4\mathbb{Z}} e_k \otimes e_{m-k} \right) (E \otimes \mathbf{1} + \mathbf{K} \otimes \mathbf{E}) \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left(E_k \otimes e_{m-k} + e_k K \otimes E_{m-k}\right) = \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left(E_k \otimes e_{m-k} + (-i)^k e_k \otimes E_{m-k}\right), \end{aligned}$$

$$\begin{aligned} \Delta(F_m) &= \Delta(e_m)\Delta(F) = \frac{1}{4} \left(\sum_{k \in \mathbb{Z}/4\mathbb{Z}} e_k \otimes e_{m-k} \right) \left(F \otimes K^{-1} + \mathbf{1} \otimes \mathbf{F}\right) \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left(F_k \otimes e_{m-k} K^{-1} + e_k \otimes F_{m-k}\right) = \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left((i)^{m-k} F_k \otimes e_{m-k} + e_k \otimes F_{m-k}\right), \end{aligned}$$

$$\begin{aligned} \Delta(P_m) &= \Delta(E_m)\Delta(F) = \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left(E_k \otimes e_{m-k} + (-i)^k e_k \otimes E_{m-k}\right) \left(F \otimes K^{-1} + \mathbf{1} \otimes \mathbf{F}\right) \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left(E_k F \otimes e_{m-k} K^{-1} + E_k \otimes F_{m-k} + (-i)^k F_k \otimes E_{m-k} K^{-1} + (-i)^k e_k \otimes E_{m-k} F\right) \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} \left((i)^{m-k} P_k \otimes e_{m-k} + E_k \otimes F_{m-k} - (-1)^k (i)^m F_k \otimes E_{m-k} + (-i)^k e_k \otimes P_{m-k}\right). \end{aligned}$$

Check of 27: We have by the antimultiplicativity of S,

$$S(e_m) = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^{mk} S(K)^k = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^{mk} K^{-k} = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^{-mk} K^k = e_{-m},$$

$$S(E_m) = S(E)S(e_m) = (-K^{-1}E)e_{-m} = -K^{-1}e_{2-m}E = -(i)^{2-m}E_{2-m} = (-i)^m E_{2-m},$$

$$S(F_m) = S(F)S(e_m) = (-FK)e_{-m} = -(i)^m F e_{-m} = -(i)^m e_{2-m} F = -(i)^m F_{2-m},$$

$$\begin{aligned} S(P_m) &= S(F)S(E)S(e_m) = (-FK)(-K^{-1}E)e_{-m} = FEe_{-m} = (EF - e_1 + e_3)e_{-m} \\ &= P_{-m} - \delta_{1,-m} e_1 + \delta_{3,-m} e_3 = P_{-m} - \delta_{3,m} e_1 + \delta_{1,m} e_3. \end{aligned}$$

Check of 28: Owing to the multiplicativity of ε, we have

$$\varepsilon(e_m) = \sum_{k\in\mathbb{Z}/4\mathbb{Z}} (i)^{mk}\varepsilon(K)^k = \sum_{k\in\mathbb{Z}/4\mathbb{Z}} (i)^{mk}\delta_{0,m},$$

whilst $\varepsilon(E_m), \varepsilon(F_m)$ and $\varepsilon(P_m)$ all vanish because they all contain the factor $\varepsilon(E)$ or $\varepsilon(F)$, $\square$

4 Adjoint representations of H_1^i and H_2^i

4.1 Adjoint representation and adjoint trace of H_1^i

Proposition 4.1 *1) We have the following values of the biregular (or adjoint) representation μ*

a	$\mu(a)e_0$	$\mu(a)e_1$	$\mu(a)E_0$	$\mu(a)E_1$	$\mu(a)F_0$	$\mu(a)F_1$	$\mu(a)C_0$	$\mu(a)C_1$
e_0	e_0	e_1	0	0	0	0	C_0	C_1
e_1	0	0	E_0	E_1	F_0	F_1	0	0
E_0	0	0	0	0	C_0+C_1	C_0+C_1	0	0
E_1	$-E_0+E_1$	E_0-E_1	0	0	0	0	0	0
F_0	0	0	$-C_0-C_1$	C_0+C_1	0	0	0	0
F_1	F_0+F_1	$-F_0+F_1$	0	0	0	0	0	0
C_0	$2\,(C_0+C_1)$	$2\,(C_0-C_1)$	0	0	0	0	0	0
C_1	0	0	0	0	0	0	0	0

2) We have the following values of the adjoint trace Tr_μ of H_1^i

a	e_0	e_1	E_0	E_1	F_0	F_1	C_0	C_1
$tr_\mu(a)$	4	4	0	0	0	0	0	0

Thus, Tr_μ has the nilradical as its kernel and passes to the semi-simple quotient as its trace.
3) The scalar product determined by the adjoint trace and the $*$-operation

$$< a, b >= \frac{1}{4}Tr_\mu(a^*b) \quad a, b \in H_1^i$$

has the only non-vanishing values $< e_0, e_0 >= 1$ and $< e_1, e_1 >= 1$, In other terms the scalar product $< .,. >$ is positive semi-definite with null-space the nilradical N_1^i.

Proof:

1) From 23, 24, and 25, we deduce the following table of elements $(id \otimes S)\Delta(a)$ of $End_{\mathbb{C}}(H_1^i) \otimes End_{\mathbb{C}}(H_1^i)$, $a \in H_1^i$

$$\begin{aligned}
(id \otimes S)\Delta(e_0) &= e_0 \otimes e_0 + e_1 \otimes e_1, \qquad (29)\\
(id \otimes S)\Delta(e_1) &= e_0 \otimes e_1 + e_1 \otimes e_0,\\
(id \otimes S)\Delta(E_0) &= E_0 \otimes e_0 + E_1 \otimes e_1 + e_0 \otimes E_1 + e_1 \otimes E_0,\\
(id \otimes S)\Delta(E_1) &= E_0 \otimes e_1 + E_1 \otimes e_0 - e_0 \otimes E_0 - e_1 \otimes E_1,\\
(id \otimes S)\Delta(F_0) &= F_0 \otimes e_0 - F_1 \otimes e_1 - e_0 \otimes F_1 + e_1 \otimes F_0,\\
(id \otimes S)\Delta(F_1) &= -F_0 \otimes e_1 + F_1 \otimes e_0 + e_0 \otimes F_0 - e_1 \otimes F_1,\\
(id \otimes S)\Delta(C_0) &= C_0 \otimes e_0 - C_1 \otimes e_1 + e_0 \otimes C_0 - e_1 \otimes C_1,\\
&\quad -E_0 \otimes F_1 + E_1 \otimes F_0 - F_0 \otimes E_1 + F_1 \otimes E_0,\\
(id \otimes S)\Delta(C_1) &= -C_0 \otimes e_1 + C_1 \otimes e_0 + e_0 \otimes C_1 - e_1 \otimes C_0,\\
&\quad -F_0 \otimes E_0 + F_1 \otimes E_1 + E_0 \otimes F_0 - E_1 \otimes F_1.
\end{aligned}$$

From these relations, the corresponding $\mu(a)x$, $x \in H_1^i$, are obtained by making $\otimes \to x$ (observe that $\mu(e_0)$ and $\mu(e_1)$ are the projections onto the even and odd part of $End_{\mathbb{C}}(H_1^i)$ for the $\mathbb{Z}/2\mathbb{Z}$-grading).

2) Results by inspection conferring 29 with 17.

3) The scalar product $< .,. >$ is given by the following table (where $< a, b >$ is plotted at the intersection of the line a and the column b), obtained by replacing the entries of 17 by their quantum traces.

	e_0	e_1	E_0	E_1	F_0	F_1	C_0	C_1
e_0	1	0	0	0	0	0	0	0
e_1	0	1	0	0	0	0	0	0
E_0	0	0	0	0	0	0	0	0
E_1	0	0	0	0	0	0	0	0
F_0	0	0	0	0	0	0	0	0
F_1	0	0	0	0	0	0	0	0
C_0	0	0	0	0	0	0	0	0
C_1	0	0	0	0	0	0	0	0

□

4.2 Adjoint representation and adjoint trace of H_2^i

Proposition 4.2 *1) The biregular representation $\mu = \lambda * \rho$ of H_2^i vanishes on $H_2^{i(1)}$ and is given on $H_2^{i(0)}$ by following table.*

a	$\mu(a)e_0$	$\mu(a)e_2$	$\mu(a)E_0$	$\mu(a)E_2$	$\mu(a)F_0$	$\mu(a)F_2$	$\mu(a)P_0$	$\mu(a)P_2$
0	$\frac{1}{4}e_0$	$\frac{1}{4}e_2$	0	0	0	0	$\frac{1}{4}P_0$	$\frac{1}{4}P_2$
e_2	0	0	$\frac{1}{4}E_0$	$\frac{1}{4}E_2$	$\frac{1}{4}F_0$	$\frac{1}{4}F_2$	0	0
E_0	$\frac{1}{4}(E_0+E_2)$	$\frac{1}{4}(E_0+E_2)$	0	0	0	0	0	0
E_2	0	0	0	0	0	0	0	0
F_0	0	0	$\frac{1}{2}P_0$	$-\frac{1}{2}P_2$	0	0	$\frac{1}{4}P_0$	$-\frac{1}{4}P_2$
F_2	$\frac{1}{4}(F_0+F_2)$	$\frac{1}{4}(F_0+F_2)$	0	0	0	0	0	0
P_0	0	0	0	0	0	0	0	0
P_2	0	0	0	0	0	0	0	0

and

a	$\mu(a)e_1$	$\mu(a)e_3$	$\mu(a)E_1$	$\mu(a)E_3$	$\mu(a)F_1$	$\mu(a)F_3$	$\mu(a)P_1$	$\mu(a)P_3$
e_0	$\frac{1}{4}e_1$	$\frac{1}{4}e_3$	0	0	0	0	$\frac{1}{4}P_1$	$\frac{1}{4}P_3$
e_2	0	0	$\frac{1}{4}E_1$	$\frac{1}{4}E_3$	$\frac{1}{4}F_1$	$\frac{1}{4}F_3$	0	0
E_0	$\frac{1}{4}(E_1+E_3)$	$\frac{1}{4}(E_1+E_3)$	0	0	$\frac{1}{4}e_1$	$-\frac{1}{4}e_3$	$-\frac{1}{4}E_1$	$\frac{1}{4}E_3$
E_2	0	0	0	0	0	0	0	0
F_0	0	0	ie_1	ie_1	0	0	$\frac{i}{4}P_1$	$-\frac{i}{4}P_3$
F_2	$-\frac{i}{4}(F_1-F_3)$	$\frac{i}{4}(F_1-F_3)$	0	0	0	0	0	0
P_0	$\frac{1}{4}(e_1+e_3)$	$\frac{1}{4}(e_1+e_3)$	0	0	0	0	0	0
P_2	0	0	0	0	0	0	0	0

We have the generic formulae

$$\left\{\begin{array}{l} \mu(e_m)e_j = \frac{1}{4}\delta_{0,m}e_j, \\ \mu(e_m)E_j = \frac{1}{4}\delta_{0,m}E_j, \\ \mu(e_m)F_j = \frac{1}{4}\delta_{2,m}F_j, \\ \mu(e_m)P_j = \frac{1}{4}\delta_{0,m}P_j, \end{array}\right. \quad \left\{\begin{array}{c} \mu(E_m)e_j = \frac{1}{4}\delta_{0,m}\left(E_{j+2}+E_j\right), \\ \mu(E_m)E_j = 0, \\ \mu(E_m)F_j = \frac{1}{4}\delta_{0,m}\left(\delta_{1,j}e_1 - \delta_{3,j}e_3\right), \\ \mu(E_m)P_j = \frac{1}{4}\delta_{0,m}\left(\delta_{1,j}E_1 - \delta_{3,j}E_3\right), \end{array}\right. \tag{30}$$

$$\left\{\begin{array}{c} \mu(F_m)e_j = \frac{1}{4}\delta_{2,m}\left((i)^j F_{j+2} + (i)^{-j} F_j\right), \\ \mu(F_m)E_j = \frac{1}{4}\delta_{0,m}\left(\left((i)^j + (i)^{-j}\right) P_j + i\left(\delta_{1,j}e_1 + \delta_{3,j}e_3\right)\right), \\ \mu(F_m)F_j = 0, \\ \mu(F_m)P_j = \frac{1}{4}\delta_{0,m}(i)^j P_j, \end{array}\right.$$

$$\left\{\begin{array}{c} \mu(P_m)e_j = \frac{1}{4}i\delta_{0,m}\left(\delta_{1,j}+\delta_{3,j}\right)\left(e_1+e_3\right), \\ \mu(P_m)E_j = -\frac{1}{4}\delta_{0,m}\left(\delta_{1,j}E_1-\delta_{3,j}E_3\right), \\ \mu(P_m)F_j = 0, \\ \mu(P_m)P_j = 0. \end{array}\right.$$

2) We have the following values of the adjoint trace Tr_μ of H_2^i.

a	e_0	e_2	E_0	E_2	F_0	F_2	C_0	C_2	e_1	e_3	E_1	E_3	F_1	F_3	P_1	P_3
$Tr_\mu(a)$	*1*	*2*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*	*0*

3) The scalar product determined by the trace and the $*$-operation

$$< a,b >= \frac{1}{4}Tr_\mu(a^*b) \quad a,b \in H_2^i$$

has the only non vanishing values $< e_0,e_0 >= 1$ and $< e_2,e_2 >= 2$. In other terms, the scalar product $< .,. >$ is positive semi definite with null-space $N_2^{i(0)} \oplus H_2^{i(1)}$.

Proof:

1) From 26 and 27 we deduce the following elements $(id\otimes S)\,\Delta(a)$ of $End_{\mathbb{C}}(H_2^i)\otimes End_{\mathbb{C}}(H_2^i)$, $a \in H_2^i$, $(id\otimes S)\,\Delta(e_m) = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}} e_k\otimes e_{k-m}$, from which the corresponding $4\mu(a)x, x \in H_2^i$, are obtained by making $\otimes \to x$, yielding 30

$$\left\{\begin{array}{c} \mu(e_m)e_j = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}} e_k e_j e_{k-m} = \frac{1}{4}\delta_{0,m}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}}\delta_{k,j}e_j = \frac{1}{4}\delta_{0,m}e_j, \\ \mu(e_m)E_j = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}} e_k E_j e_{k-m} = \frac{1}{4}\delta_{0,m}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}}\delta_{k,j}E_j = \frac{1}{4}\delta_{0,m}E_j, \\ \mu(e_m)F_j = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}} e_k F_j e_{k-m} = \frac{1}{4}\delta_{k+2,k-m}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}}\delta_{k,j}F_j = \frac{1}{4}\delta_{2,m}F_j, \\ \mu(e_m)P_j = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}} e_k P_j e_{k-m} = \frac{1}{4}\delta_{0,m}\sum\limits_{k\in\mathbb{Z}/2\mathbb{Z}}\delta_{k,j}p_j = \frac{1}{4}\delta_{0,m}P_j. \end{array}\right.$$

The derivation of the other formulae is obtained via multiplicativity of μ. Combining the latter with

$$\left\{\begin{array}{c} \mu(E)(e_j) = E_{j+2}+E_j, \\ \mu(E)(E_j) = 0, \\ \mu(E)(F_j) = -\left(\delta_{1,j}e_1-\delta_{3,j}e_3\right), \\ \mu(E)(P_j) = -\left(\delta_{1,j}E_1-\delta_{3,j}E_3\right), \end{array}\right. \quad \left\{\begin{array}{c} \mu(F)(e_j) = (i)^j\,F_{j+2}+(i)^{-j}\,F_j, \\ \mu(F)(E_j) = -\left((i)^j+(i)^{-j}\right)P_j-(i)^j\left(\delta_{1,j}e_1-\delta_{3,j}e_3\right), \\ \mu(F)(F_j) = 0, \\ \mu(F)(P_j) = -i\left(E_1+E_3\right), \end{array}\right. \tag{31}$$

$$\left\{\begin{array}{c} \mu(P)(e_j) = i\left(\delta_{1,j}+\delta_{3,j}\right)\left(e_1+e_3\right), \\ \mu(P)(E_j) = 2i\left(E_1+E_3\right), \\ \mu(P)(F_j) = 0, \\ \mu(P)(P_j) = 0. \end{array}\right. \tag{32}$$

they are checked as follows: from 2 we have $(id\otimes S)\,\Delta(E)=\mathbf{E}\otimes\mathbf{1}+\mathbf{K}\otimes\mathbf{K^{-1}E}$ and $(id\otimes S)\,\Delta(F)=\mathbf{F}\otimes\mathbf{K^{-1}}-\mathbf{1}\otimes\mathbf{FK}$. Hence we havefor $x \in H_2^i$

$$\begin{aligned} \mu(E)(x) &= Ex+KxK^{-1}E, \\ \mu(F)(x) &= FxK^{-1}+xKF, \end{aligned}$$

and thus

$$\begin{aligned} \mu(E)e_j &= Ee_j+Ke_jK^{-1}E = E_{j+2}+E_j, \\ \mu(E)E_j &= EE_j+KE_jK^{-1}E = 0, \\ \mu(E)F_j &= EF_j+KF_jK^{-1}E = P_j-e_j\left(EF+e_1-e_3\right) = -\left(\delta_{1,j}e_1-\delta_{3,j}e_3\right), \\ \mu(E)P_j &= EP_j+KP_jK^{-1}E = e_jEFE = -e_j\left(e_1-e_3\right)E = -\left(\delta_{1,j}E_1-\delta_{3,j}E_3\right), \end{aligned}$$

and

$$\begin{aligned}
\mu(F)e_j &= Fe_jK^{-1} + e_jKF = (i)^j F_{j+2} + (i)^{-j} F_j, \\
\mu(F)E_j &= FE_jK^{-1} + E_jKF = (i)^{j+2} e_j (EF + e_1 - e_3) - (i)^{-j} P_j, \\
&= -\left((i)^j + (i)^{-j}\right) P_j - (i)^j (\delta_{1,j}e_1 - \delta_{3,j}e_3), \\
\mu(F)F_j &= FF_jK^{-1} + F_jKF = 0, \\
\mu(F)P_j &= FP_jK^{-1} + P_jKF = (i)^j e_{j+2}FEF = (i)^j e_{j+2} (e_1 - e_3) F, \\
&= (i)^j (\delta_{3,j}F_1 - \delta_{1,j}F_3) = -i (E_1 + E_3),
\end{aligned}$$

further

$$\begin{aligned}
\mu(P)e_j &= \mu(E)\mu(F)e_j = \mu(E) \left((i)^j F_{j+2} + (i)^{-j} F_j\right), \\
&= -(i)^j (\delta_{1,j+2}e_1 - \delta_{3,j+2}e_3) - (i)^{-j} (\delta_{1,j}e_1 - \delta_{3,j}e_3), \\
&= -(i)^j (\delta_{3,j}e_1 - \delta_{1,j}e_3) - (i)^{-j} (\delta_{1,j}e_1 - \delta_{3,j}e_3), \\
&= i\delta_{3,j}e_1 + i\delta_{1,j}e_3 + i\delta_{1,j}e_1 + i\delta_{3,j}e_3 = i (\delta_{1,j} + \delta_{3,j}) (e_1 + e_3), \\
\mu(P)E_j &= \mu(E)\mu(F)E_j = \mu(E) \left(-\left((i)^j + (i)^{-j}\right) P_j - (i)^j (\delta_{1,j}e_1 - \delta_{3,j}e_3)\right), \\
&= \left((i)^j + (i)^{-j}\right) (\delta_{1,j}E_1 - \delta_{3,j}E_3) + i\mu(E) (e_1 + e_3), \\
&= 2i (E_1 + E_3). \\
\mu(p)F_j &= \mu(E)\mu(F)F_j = 0
\end{aligned}$$

We now compute successively

$$\begin{aligned}
\mu(E_m)e_j &= \mu(e_m)\mu(E)e_j = \mu(e_m) (E_{j+2} + E_j) = \frac{1}{4}\delta_{0,m} (E_{j+2} + E_j), \\
\mu(E_m)E_j &= \mu(E)\mu(e_{m+2})E_j = \frac{1}{4}\delta_{0,m+2}\mu(E)(E_j) = 0, \\
\mu(E_m)F_j &= \mu(E)\mu(e_{m+2})F_j = \frac{1}{4}\delta_{2,m+2}\mu(E)(F_j) = \frac{1}{4}\delta_{0,m} (\delta_{1,j}e_1 - \delta_{3,j}e_3), \\
\mu(E_m)P_j &= \mu(e_m)\mu(E)P_j = \mu(E), \\
&= -\mu(e_m) (\delta_{1,j}E_1 - \delta_{3,j}E_3) = -\frac{1}{4}\delta_{0,m} (\delta_{1,j}E_1 - \delta_{3,j}E_3),
\end{aligned}$$

and

$$\begin{aligned}
\mu(F_m)e_j &= \mu(e_m)\mu(F)e_j = \mu(e_m) \left((i)^j F_{j+2} + (i)^{-j} F_j\right) = \frac{1}{4}\delta_{2,m} \left((i)^j F_{j+2} + (i)^{-j} F_j\right), \\
\mu(F_m)E_j &= \mu(e_m)\mu(F)E_j = \mu(e_m) \left(-\left((i)^j + (i)^{-j}\right) P_j - (i)^j (\delta_{1,j}e_1 - \delta_{3,j}e_3)\right), \\
&= -\frac{1}{4}\delta_{0,m} \left(\left((i)^j + (i)^{-j}\right) P_j + (i)^j (\delta_{1,j}e_1 - \delta_{3,j}e_3)\right), \\
\mu(F_m)F_j &= \mu(e_m)\mu(F)F_j = 0, \\
\mu(F_m)P_j &= \mu(e_m)\mu(F)P_j = (i)^j \mu(e_m)P_j = \frac{1}{4}\delta_{0,m} (i)^j P_j,
\end{aligned}$$

and

$$\mu(P_m)e_j = \mu(e_m)\mu(P)e_j = i\mu(e_m) (\delta_{1,j} + \delta_{3,j}) (e_1 + e_3),$$

$$
\begin{aligned}
&= \frac{1}{4} i\delta_{0,m}\left(\delta_{1,j}+\delta_{3,j}\right)\left(e_1+e_3\right), \\
\mu(P_m)E_j &= \mu(e_m)\mu(P)E_j = -\mu(e_m)\left(\delta_{1,j}E_1-\delta_{3,j}E_3\right) = -\frac{1}{4}\delta_{0,m}\left(\delta_{1,j}E_1-\delta_{3,j}E_3\right), \\
\mu(P_m)F_j &= \mu(E)\mu(e_{m+2})F_j = 0, \\
\mu(P_m)P_j &= \mu(e_m)\mu(E)\mu(F)P_j = 0.
\end{aligned}
$$

□

5 Idempotents, automorphisms and real forms of H_1^i

Lemma 5.1 *1) There is a unique $\kappa \in Aut(H_1^i)$ (the flip) such that*

$$
\begin{cases} \kappa(K) = -K, \\ \kappa(E) = E, \\ \kappa(F) = F. \end{cases}
$$

2) κ is an involution performing the exchanges $e_0 \leftrightarrow e_1, E_0 \leftrightarrow E_1, F_0 \leftrightarrow F_1$ and $C_0 \leftrightarrow C_1$.
3) One has $\kappa \circ S \circ \kappa = S^{-1}$.

Proof:
1) and 2) : immediate from 17.
3) straightforward from $S^{-1}(K) = K^{-1}$, $S^{-1}(E) = K^{-1}E$ and $S^{-1}(K) = FK$. □

5.1 Idempotents of H_1^i

Proposition 5.1 *1) The idempotents of H_1^i are*
-the element 0 of rank 0,
*-the unit **1** of rank 8,*
-a continous famuly of rank 4 idempotents

$$
\begin{cases} e_{0,\beta,\gamma,\delta,\eta} = e_0 + \beta E_0 + \gamma F_0 + \delta E_1 + \eta F_1 + (\beta\eta + \delta\gamma)(C_1 - C_0), \\ e_{1,\beta,\gamma,\delta,\eta} = e_1 + \beta E_0 + \gamma F_0 + \delta E_1 + \eta F_1 + (\beta\eta + \delta\gamma)(C_1 - C_0), \end{cases}
$$

indexed by the parameters $\beta, \gamma, \delta, \eta \in \mathbb{C}$.
2) Consequently the automorphisms ϕ of H_1^i are of either the two types:
a)fulfilling $\phi(e_0) = e_{0,\beta,\gamma,\delta,\eta}, \phi(e_1) = e_{1,-\beta,-\gamma,-\delta,-\eta}$,
b)fulfilling $\phi(e_0) = e_{1,\beta,\gamma,\delta,\eta}, \phi(e_1) = e_{0,-\beta,-\gamma,-\delta,-\eta}$.

Proof:
1) Every idempotent e of $H_1^i = F_1^i \oplus N_1^i$ (F_1^i a subalgebra, N_1^i an ideal) decomposes as $e = e' + e''$ with $e'^2 = e', e'e'' + e''e' + e''^2 = e''$. We search e'
$e' = \alpha e_0 + \beta e_1 = e'^2 = \alpha^2 e_0 + \beta^2 e_1$ yields $\alpha(\alpha - 1) = \beta(\beta - 1) = 0$, hence e' either $0, e_0, e_1$, or $e_0 + e_1$.
We search $e'' = x_0E_0 + y_0F_0 + v_0C_0 + x_1E_1 + y_1F_1 + v_1C_1$ with $e'' = (x_0y_1 + x_1y_0)(C_0 + C_1)$
-To $e' = 0, e''^2 = e''$ yields $x_0 = y_0 = x_1 = y_1 = 0, v_0 = v_1 = x_0y_1 + x_1y_0 = 0$ thus $e'' = 0$
-To $e' = \mathbf{1}, \mathbf{2e''} + \mathbf{e''^2} = \mathbf{e''}, \mathbf{e''^2} = -\mathbf{e''}$ yields $x_0 = y_0 = x_1 = y_1 = 0$, $v_0 = v_1 = -(x_0y_1 + x_1y_0)\, x_0y_1 + x_1y_0 = 0$ thus $e'' = 0$
-To $e' = e_0$ since $e_0e'' + e_0e'' = x_0E_0 + y_0F_0 + v_0C_0 + x_1E_1 + y_1F_1 + v_1C_1 = e'' + +v_0C_0 - v_1C_1$
$e_0e'' + e_0e'' = e''$ yields $v_1 = -v_0 = x_0y_1 + x_1y_0$, x_0, y_0, x_1, y_1 remaining arbitrary whence

$e = e_0 + x_0 E_0 + y_0 F_0 + x_1 E_1 + y_1 F_1 + (x_0 y_1 + x_1 y_0)(C_0 - C_1)$. The flip then yields by exchange symmetry $0 \leftrightarrow 1$ the idempotent $e_1 + x_0 E_0 + y_0 F_0 + x_1 E_1 + y_1 F_1 + (x_0 y_1 + x_1 y_0)(C_0 - C_1)$

Rank of $e_{0,\beta,\gamma,\delta,\eta}$: $h = a_0 e_0 + X_0 E_0 + Y_0 F_0 + c_0 C_0 + a_1 e_1 + X_1 E_1 + Y_1 F_1 + c_1 C_1$ fulfills $e_{0,\beta,\gamma,\delta,\eta} h = h$ iff one has the relations $a_1 = 0$, $\eta X_0 + \delta Y_0 = (\beta\eta + \delta\gamma) a_0$ (automatic), $X_1 = \delta a_0$, $Y_1 = \eta a_0$. Thus in the two occurring cases $\delta\eta = 0$ and $\delta \neq 0, \eta \neq 0$, one has rank 4.

2) Obvious from (1) since automorphisms turn an idempotent into an idempotent of the same rank, Morover, φ must be bijective and satisfy $\varphi(e_1) + \varphi(e_0) = \mathbf{1}$. $\square$

5.2 Automorphisms of H_1^i

Proposition 5.2 *1) The set $Aut(H_1^i)$ of automorphisms of H_1^i coinciding with $Aut(N_1^i)$ consists of elements of the two types (whose respective sets will be denoted $Aut_I(H_1^i)$ and $Aut_{II}(H_1^i)$.*

Type I :

$$\begin{aligned} \varphi(e_0) &= e_0 + \beta E_0 + \gamma F_0 + \delta E_1 + \eta F_1 + (\beta\eta + \gamma\delta)(C_1 - C_0), \\ \varphi(E_0) &= \mu_0 E_0 + \nu_0 F_0 + (\delta\nu_0 + \eta\mu_0)(C_1 - C_0), \\ \varphi(F_0) &= \sigma_0 E_0 + \tau_0 F_0 + (\delta\tau_0 + \eta\sigma_0)(C_1 - C_0), \\ \varphi(C_0) &= \lambda C_0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} \varphi(e_1) &= e_1 - \beta E_0 - \gamma F_0 - \delta E_1 - \eta F_1 - (\beta\eta + \gamma\delta)(C_1 - C_0), \\ \varphi(E_1) &= \mu_1 E_1 + \nu_1 F_1 + (\beta\nu_1 + \gamma\mu_1)(C_1 - C_0), \\ \varphi(F_1) &= \sigma_1 E_1 + \tau_1 F_1 + (\beta\tau_1 + \gamma\sigma_1)(C_1 - C_0), \\ \varphi(C_1) &= \lambda C_1, \end{aligned} \tag{34}$$

for constants $\beta, \gamma, \delta, \eta, \lambda, \mu_0, \nu_0, \sigma_0, \tau_0, \mu_1, \nu_1, \sigma_1, \tau_1 \in \mathbb{C}$ constrained by

$$\begin{cases} 1)\mu_0\nu_1 + \nu_0\mu_1 = 0, \\ 2)\sigma_0\tau_1 + \tau_0\sigma_1 = 0, \\ 3)\lambda = \mu_0\tau_1 + \nu_0\sigma_1 = \sigma_0\nu_1 + \tau_0\mu_1, \end{cases} \quad \begin{cases} 4)\mu_0\tau_0 - \nu_0\sigma_0 \neq 0, \\ 5)\mu_1\tau_1 - \nu_0\sigma_0 \neq 0, \\ 6)\lambda \neq 0. \end{cases} \tag{35}$$

Type II : Product $\varphi\kappa$ (or for that matter $\kappa\varphi$) with φ of the preceding type I

$$\begin{aligned} \varphi(e_0) &= e_1 + \beta E_1 + \gamma F_1 + \delta E_0 + \eta F_0 + (\beta\eta + \gamma\delta)(C_0 - C_1), \\ \varphi(E_0) &= \mu_0 E_1 + \nu_0 F_1 + (\delta\nu_0 + \eta\mu_0)(C_0 - C_1), \\ \varphi(F_0) &= \sigma_0 E_1 + \tau_0 F_1 + (\delta\tau_0 + \eta\sigma_0)(C_0 - C_1), \\ \varphi(C_0) &= \lambda C_1, \end{aligned}$$

and

$$\begin{aligned} \varphi(e_1) &= e_0 - \beta E_1 - \gamma F_1 - \delta E_0 - \eta F_0 - (\beta\eta + \gamma\delta)(C_0 - C_1), \\ \varphi(E_1) &= \mu_1 E_0 + \nu_1 F_0 + (\beta\nu_1 + \gamma\mu_1)(C_0 - C_1), \\ \varphi(F_1) &= \sigma_1 E_0 + \tau_1 F_0 + (\beta\tau_1 + \gamma\sigma_1)(C_0 - C_1), \\ \varphi(C_1) &= \lambda C_0, \end{aligned}$$

with constants constrained as in 35*. Observe that these constraints, as well as the whole structure of H_1^i, is invariant under the flip.*

2) $Aut_I(H_1^i)$ is a normal subgroup of $Aut(H_1^i)$.

3) The subgroup $Aut(H_1^i) = Aut_I(H_1^i) \oplus Aut_{II}(H_1^i)$ of $Aut_I(H_1^i)$ specified by $\beta = \gamma = \delta = \eta = 0$, consists of the 4×4 matrices

$$M = \begin{pmatrix} \mu_0 & \nu_0 & 0 & 0 \\ \sigma_0 & \tau_0 & 0 & 0 \\ 0 & 0 & \mu_1 & \nu_1 \\ 0 & 0 & \sigma_1 & \tau_1 \end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

leaving stable the set of bilinear forms with vanishing sum of elements of their second diagonal. Specifically, for each matrix of the form

$$G=\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & d \\ a & b & 0 & 0 \\ b & d & 0 & 0 \end{pmatrix} \qquad a,b,c,d\in\mathbb{C},$$

M^tGM *is a matrix of the same type.*

4) The constraints 35 *entail the relations*

$$\begin{cases} \nu_0\tau_1+\tau_0\nu_1=0, \\ \mu_0\sigma_1+\sigma_0\mu_1=0. \end{cases} \tag{36}$$

Note that (2) answers the natural question why the constraints 35 *propagate through matrix products. Indeed it is not clear a priori that* $Aut_I(H_1^i)$ *characterized as in (1) is multiplicative. Note also how two symmetries permute (either respecting or exchanging) formula, the flip* $(0\leftrightarrow 1)$ *and the symmetry* $(E\leftrightarrow F,\mu\leftrightarrow\tau,\nu\leftrightarrow\sigma)$.

<u>Proof</u>:

1) Exhausting the constraints stemming from multiplicativity of φ, in view of the multiplication table 17, we have the implications

$$\begin{aligned}
\varphi(e_0E_0) &= \varphi(e_0)\varphi(E_0)\Rightarrow \mu_1'=\nu_1'=0,\rho_1'=\delta\nu_0+\eta\mu_0,\\
\varphi(e_0F_0) &= \varphi(e_0)\varphi(F_0)\Rightarrow \sigma_1'=\tau_1'=0,\omega_1'=\delta\tau_0+\eta\sigma_0,\\
\varphi(E_0e_1) &= \varphi(E_0)\varphi(e_1)\Rightarrow \mu_1'=\nu_1'=0,\rho_0'=-\left(\delta\nu_0+\eta\mu_0\right),\\
\varphi(F_0e_1) &= \varphi(F_0)\varphi(e_1)\Rightarrow \sigma_1'=\tau_1'=0,\omega_1'=-\left(\delta\tau_0+\eta\sigma_0\right),\\
\varphi(e_1F_1) &= \varphi(e_1)\varphi(F_1)\Rightarrow \sigma_0'=\tau_0'=0,\omega_0'=-\left(\beta\tau_1+\gamma\sigma_1\right),\\
\varphi(E_1e_0) &= \varphi(E_1)\varphi(e_0)\Rightarrow \mu_0'=\nu_0'=0,\rho_1=\beta\nu_1+\gamma\mu_1,\\
\varphi(F_1e_0) &= \varphi(F_1)\varphi(e_0)\Rightarrow \sigma_0'=\tau_0'=0,\omega_1=\beta\tau_1+\gamma\sigma_1,\\
\varphi(F_0E_1) &= \varphi(F_0)\varphi(E_1)\Rightarrow \varphi(C_0)=\left(\sigma_0\nu_1+\tau_0\mu_1\right)C_0,\\
\varphi(E_0F_1) &= \varphi(E_0)\varphi(F_1)\Rightarrow \varphi(C_0)=\left(\mu_0\tau_1+\nu_0\sigma_1\right)C_0,\\
\varphi(E_1F_0) &= \varphi(E_1)\varphi(F_0)\Rightarrow \varphi(C_1)=\left(\mu_1\tau_0+\nu_1\sigma_0\right)C_1,\\
\varphi(F_1E_0) &= \varphi(F_1)\varphi(E_0)\Rightarrow \varphi(C_1)=\left(\mu_0\tau_1+\nu_0\sigma_1\right)C_1,\\
\varphi(E_0E_1) &= \varphi(E_0)\varphi(E_1)\Rightarrow \mu_0\nu_1+\nu_0\mu_1=0,\\
\varphi(F_0F_1) &= \varphi(F_0)\varphi(F_1)\Rightarrow \sigma_0\tau_1+\tau_0\sigma_1=0.
\end{aligned}$$

2) Let $\varphi\in Aut_I(H_1^i)$, $\psi=\varphi_1\kappa\in Aut_{II}(H_1^i)$, $\varphi_1\in Aut_I(H_1^i)$, since $\psi^{-1}\varphi\psi=\kappa^{-1}\varphi_1^{-1}\varphi\varphi_1\kappa$, it suffices to prove that $ad\kappa$ leaves $Aut_I(H_1^i)$ stable, now with φ as in 33,34, one has $\kappa^{-1}\varphi\kappa=\varphi'$, φ' of the form 33 and 34, with $\beta'=-\delta,\gamma'=-\eta,\delta'=-\beta,\eta'=-\gamma$, $\mu_0'=\mu_1,\nu_0'=\nu_1$, $\sigma_0'=\sigma_1,\tau_0'=\tau_1,\mu_1'=\mu_0,\nu_1'=\nu_0,\sigma_1'=\sigma_0,\tau_1'=\tau_0$.

3) Follows from

$$\begin{aligned}
M^tGM &= \begin{pmatrix} \mu_0 & \sigma_0 & 0 & 0 \\ \nu_0 & \tau_0 & 0 & 0 \\ 0 & 0 & \mu_1 & \sigma_1 \\ 0 & 0 & \nu_1 & \tau_1 \end{pmatrix}\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & d \\ a & -b & 0 & 0 \\ b & d & 0 & 0 \end{pmatrix}\begin{pmatrix} \mu_0 & \nu_0 & 0 & 0 \\ \sigma_0 & \tau_0 & 0 & 0 \\ 0 & 0 & \mu_1 & \nu_1 \\ 0 & 0 & \sigma_1 & \tau_1 \end{pmatrix}\\
&= \begin{pmatrix} 0 & 0 & a\mu_0-b\sigma_0 & b\mu_0+d\sigma_0 \\ 0 & 0 & a\nu_0-b\tau_0 & -b\nu_0+d\tau_0 \\ a\mu_1+b\sigma_1 & -b\mu_1+d\sigma_1 & 0 & 0 \\ a\nu_1+b\tau_1 & -b\nu_1+d\tau_1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mu_0 & \nu_0 & 0 & 0 \\ \sigma_0 & \tau_0 & 0 & 0 \\ 0 & 0 & \mu_1 & \nu_1 \\ 0 & 0 & \sigma_1 & \tau_1 \end{pmatrix}.
\end{aligned}$$

Thus, the symmetric matrix M^tGM will be of the form

$$\begin{pmatrix} 0 & 0 & a' & b' \\ 0 & 0 & -b' & d \\ a' & -b' & 0 & 0 \\ b' & d' & 0 & 0 \end{pmatrix}$$

for $a', b', c', d' \in \mathbb{C}$, whenever the sum $X + Y$ for its entries X, Y located as

$$\begin{pmatrix} . & . & . & X \\ . & . & Y & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix}$$

vanishes. The latter are

$$\begin{cases} X = (a\mu_0 - b\sigma_0)\,\mu_1 + (b\mu_0 + d\sigma_0)\,\nu_1, \\ Y = (a\nu_0 - b\tau_0)\,\mu_1 + (b\nu_0 + d\tau_0)\,\nu_1, \end{cases}$$

whenever the vanishing of

$$X + Y = a\,(\nu_1\mu_0 + \mu_1\nu_0) + b\,(\tau_1\mu_0 + \sigma_1\nu_0 - \nu_1\sigma_0 - \mu_1\tau_0) + d\,(\tau_1\sigma_0 + \sigma_1\tau_0)$$

expresses the constraints in 35.

4) Multiplying both sides of 35-(3) by $\tau_1\nu_1$ yield using 35-(1) and 35-(2) yields

$$-\nu_0\tau_1 D_1 = \tau_0\nu_1 D_1,$$

with $D_1 = \mu_1\tau_1 - \nu_1\sigma_1$ assumed not to vanish thus implying relation 36 $\square$

5.2.1 The group $Int(H_1^i)$ of inner automorphisms of H_1^i

Proposition 5.3 *1) The element $h = a_0e_0 + X_0E_0 + Y_0F_0 + c_0C_0 + a_1e_1 + X_1E_1 + Y_1F_1 + c_1C_1 \in H_1^i$ indexed by $a_0, X_0, Y_0, c_0, a_1, X_1, Y_1, c_1 \in \mathbb{C}$, is invertible iff $a_0a_1 \neq 0$. Its inverse is then*

$$\begin{aligned} h^{-1} \;=\; & \frac{1}{a_0}e_0 - a_0a_1\,(X_0E_0 + Y_0F_0) + \frac{1}{a_0}\left(\frac{P}{a_0a_1} - \frac{1}{a_0}c_0\right)C_0, \\ & + \frac{1}{a_1}e_1 - a_0a_1\,(X_1E_1 + Y_1F_1) + \frac{1}{a_1}\left(\frac{P}{a_0a_1} - \frac{1}{a_1}c_1\right)C_1, \end{aligned} \tag{37}$$

where $P = X_0Y_1 + X_1Y_0$.

2) The matrix of $ad(h)$ then reads

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{X_0}{a_1} & \frac{a_0}{a_1} & 0 & 0 & \frac{X_0}{a_1} & 0 & 0 & 0 \\ -\frac{Y_0}{a_1} & 0 & \frac{a_0}{a_1} & 0 & \frac{Y_0}{a_1} & 0 & 0 & 0 \\ \frac{P}{a_0a_1} & -\frac{Y_1}{a_1} & -\frac{X_1}{a_1} & 1 & -\frac{P}{a_0a_1} & \frac{Y_0}{a_0} & \frac{X_0}{a_0} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{X_1}{a_0} & 0 & 0 & 0 & -\frac{X_1}{a_0} & \frac{a_1}{a_0} & 0 & 0 \\ \frac{Y_1}{a_0} & 0 & 0 & 0 & -\frac{Y_1}{a_0} & 0 & \frac{a_1}{a_0} & 0 \\ -\frac{P}{a_0a_1} & \frac{Y_1}{a_1} & \frac{X_1}{a_1} & 0 & \frac{P}{a_0a_1} & -\frac{Y_0}{a_0} & -\frac{X_0}{a_0} & 1 \end{pmatrix} \tag{38}$$

in the basis $(e_0, E_0, F_0, C_0, e_1, E_1, F_1, C_1)$.

This displays the four-parametric group $Int(H_1^i) \subset Aut_I(H_1^i)$ of inner automophisms of H_1^i. The relationship with the parametrization of the full $Aut_I(H_1^i)$ is as follows.

$$\begin{cases} \beta = -\frac{X_0}{a_1}, \\ \gamma = -\frac{Y_0}{a_1}, \\ \delta = \frac{X_1}{a_0}, \\ \eta = \frac{Y_1}{a_0}, \end{cases} \quad \begin{cases} \mu_0 = \frac{a_0}{a_1}, \\ \nu_0 = 0, \\ \sigma_0 = 0, \\ \tau_0 = \frac{a_0}{a_1}, \end{cases} \quad \begin{cases} \mu_1 = \frac{a_1}{a_0}, \\ \nu_1 = 0, \\ \sigma_1 = 0, \\ \tau_1 = \frac{a_1}{a_0}. \end{cases} \tag{39}$$

Proof:

Let $h' = a_0' e_0 + X_0' E_0 + Y_0' F_0 + c_0' C_0 + a_1' e_1 + X_1' E_1 + Y_1' F_1 + c_1' C_1 \in H_1^i$, we have $hh' = \mathbf{1}$ iff

$$\left\{ \begin{array}{c} a_0 a_0' = 1, \\ a_0 X_0' + a_1' X_0 = 0, \\ a_0 Y_0' + a_1' Y_0 = 0, \\ a_0 c_0' + c_0 a_0' + X_0 Y_1' + Y_0 X_1' = 0, \end{array} \right. \qquad \left\{ \begin{array}{c} a_1 a_1' = 1, \\ a_0' X_1 + a_1 X_1' = 0, \\ a_1 Y_1' + a_0' Y_1 = 0, \\ a_1 c_1' + c_1 a_1' + X_1 Y_0' + Y_1 X_0' = 0, \end{array} \right.$$

hence 37 is proved. With λ (resp. ρ) the regular representation (resp. antirepresentation), we have

$$matrix\ of\ \lambda(h) = \begin{array}{cc} & \begin{array}{cccccccc} e_0 & E_0 & F_0 & C_0 & e_1 & E_1 & F_1 & C_1 \end{array} \\ \begin{array}{c} e_0 \\ E_0 \\ F_0 \\ C_0 \\ e_1 \\ E_1 \\ F_1 \\ C_1 \end{array} & \begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 & X_0 & 0 & 0 & 0 \\ 0 & 0 & a_0 & 0 & Y_0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & a_0 & 0 & Y_0 & X_0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 \\ Y_1 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & Y_1 & X_1 & 0 & c_1 & 0 & 0 & a_1 \end{pmatrix} \end{array},$$

and

$$matrix\ of\ \rho(h') = \begin{array}{cc} & \begin{array}{cccccccc} e_0 & E_0 & F_0 & C_0 & e_1 & E_1 & F_1 & C_1 \end{array} \\ \begin{array}{c} e_0 \\ E_0 \\ F_0 \\ C_0 \\ e_1 \\ E_1 \\ F_1 \\ C_1 \end{array} & \begin{pmatrix} a_0' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_0' & a_1' & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_0' & 0 & a_1' & 0 & 0 & 0 & 0 & 0 \\ c_0' & Y_1' & X_1' & a_0' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_1' & a_1' & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_1' & 0 & a_1' & 0 \\ 0 & 0 & 0 & 0 & c_1' & Y_0' & X_0' & a_1' \end{pmatrix} \end{array}.$$

The matrix product $\lambda(h)\rho(h')$ then yields relation 38 □

5.2.2 The subgroup $Aut^S(H_1^i)$ of automorphisms of H_1^i commuting with the antipode S

Proposition 5.4 *The $\varphi \in Aut^S(H_1^i)$ are of the following two types, corresponding to type $Aut_I(H_1^i)$ and type $Aut_{II}(H_1^i)$ with $\mu, \nu, \sigma, \tau \in \mathbb{C}$ such that $\mu\nu - \sigma\tau \neq 0$.*

1) The $\varphi \in Aut_I^S(H_1^i)$ are as follows

$$\left\{ \begin{array}{c} \varphi(e_0) = e_0, \\ \varphi(E_0) = \mu E_0 + \nu F_0, \\ \varphi(F_0) = \sigma E_0 + \tau F_0, \\ \varphi(C_0) = (\mu\tau - \sigma\nu)\, C_0, \end{array} \right. \quad \left\{ \begin{array}{c} \varphi(e_1) = e_1, \\ \varphi(E_1) = \mu E_1 - \nu F_1, \\ \varphi(F_1) = -\sigma E_1 + \tau F_1, \\ \varphi(C_1) = (\mu\tau - \sigma\nu)\, C_1. \end{array} \right.$$

Their action on the generators is given by

$$\left\{ \begin{array}{l} \varphi(K) = K, \\ \varphi(E) = \mu E + \nu K F, \\ \varphi(F) = \sigma K E + \tau F. \end{array} \right. \tag{40}$$

2) The $\varphi \in Aut_{II}^S(H_1^i)$ are as follows

$$\left\{ \begin{array}{c} \varphi(e_0) = e_1, \\ \varphi(E_0) = \mu E_1 + \nu F_1, \\ \varphi(F_0) = \sigma E_1 + \tau F_1, \\ \varphi(C_0) = (\mu\tau - \sigma\nu)\, C_1, \end{array} \right. \quad \left\{ \begin{array}{c} \varphi(e_1) = e_0, \\ \varphi(E_1) = -\mu E_0 + \nu F_0, \\ \varphi(F_1) = \sigma E_0 - \tau F_0, \\ \varphi(C_1) = (\mu\tau - \sigma\nu)\, C_0. \end{array} \right.$$

Their action on the generators is given by

$$\left\{\begin{array}{c}\varphi(K)=-K,\\ \varphi(E)=-\mu KE+\nu F,\\ \varphi(F)=\sigma E-\tau KF.\end{array}\right. \tag{41}$$

Proof:

We confer 14 with the action of S which we recall.

a	e_0	E_0	F_0	C_0	e_1	E_1	F_1	C_1
$S(a)$	e_0	E_1	$-F_1$	C_0	e_1	$-E_0$	F_0	C_1

1) The requirement that the action of $\varphi\circ S$ and $S\circ\varphi$ be the same,

-on e_0 yields $\beta=\gamma=\delta=\eta=0$,

-on F_0 or F_1 yields $\sigma_0=-\sigma_1$ and $\tau_0=\tau_1$,

-on C_0 is automatic.

The expression 39 immediately follows from 38 and from the facts that $E_0+E_1=E, F_0+F_1=F, E_0-E_1=KE, F_0-F_1=KF$.

2) The requirement that the action of $\varphi\circ S$ and $S\circ\varphi$ be the same,

-on e_1 yields $\beta=\gamma=\delta=\eta=0$,

-on F_1 or F_0 yields $\sigma_0=-\sigma_1$ and $\tau_0=\tau_1$,

-on C_1 is automatic, □

5.2.3 The Hopf automorphisms of H_1^i

Corollary 5.1 *Each element of $Aut_I^S(H_1^i)$ and none of $Aut_{II}^S(H_1^i)$ is a Hopf automorphism.*

Proof:

We check that $\varphi\in Aut_I^S(H_1^i)$ is coalgebra morphism. By the multiplicativity of Δ, it suffices to address the generators. We have, by 40,

$$\begin{aligned}
\Delta(\varphi(K)) &= \Delta(K)=K\otimes K=\varphi(K)\otimes\varphi(K),\\
\Delta(\varphi(E)) &= \Delta\left(\mu E+\nu KF\right)=\mu\left(E\otimes\mathbf{1}+\mathbf{K}\otimes\mathbf{E}\right)+\nu\left(K\otimes K\right)\left(F\otimes K^{-1}+\mathbf{1}\otimes\mathbf{F}\right),\\
&= \left(\mu E+\nu KF\right)\otimes\mathbf{1}+\mathbf{K}\otimes\left(\mu\mathbf{E}+\nu\mathbf{KF}\right)=\varphi(\mathbf{E})\otimes\varphi(\mathbf{1})+\varphi(\mathbf{K})\otimes\varphi\left(\mathbf{F}\right),\\
\Delta(\varphi(F)) &= \Delta\left(\sigma KE+\tau F\right)=\sigma\left(K\otimes K\right)\left(E\otimes\mathbf{1}+\mathbf{K}\otimes\mathbf{E}\right)+\tau\left(F\otimes K^{-1}+\mathbf{1}\otimes\mathbf{F}\right),\\
&= \left(\sigma KE+\tau F\right)\otimes K^{-1}+\mathbf{1}\otimes\left(\sigma\mathbf{KE}+\tau\mathbf{F}\right)=\varphi(\mathbf{F})\otimes\varphi(\mathbf{K^{-1}})+\varphi(\mathbf{1})\otimes\varphi(\mathbf{F}),
\end{aligned}$$

whereas, for $\varphi\in Aut_{II}^S(H_1^i)$, we have by 41,

$$\Delta(\varphi(K))=-\Delta(K)=-K\otimes K\neq\varphi(K)\otimes\varphi(K)=K\otimes K.$$

□

Lemma 5.2 *The identity on the generators K,E,F extends uniquely to a Hopf $*$-operation I of H_1^i, whose action is given as follows.*

a	e_0	E_0	F_0	C_0	e_1	E_1	F_1	C_1
$I(a)$	e_0	E_1	F_1	C_0	e_1	E_0	F_0	C_1

Proof:

The defining 12 and 13 are obviously respected, as well as the definition relations of the Hopf structure stated on the generators. $\square$

Proposition 5.5 *1) The semi-Hopf $*$-operations of H_1^i are the of the following two types, corresponding to the above type I and type II automorphisms.*

Type I: $\Gamma = I \circ \varphi, \varphi \in Aut_I^S(H_1^i)$ is given by

$$\left\{ \begin{array}{c} \Gamma(e_0) = e_0, \\ \Gamma(E_0) = \alpha E_1 + \beta F_1, \\ \Gamma(F_0) = \gamma E_1 + \delta F_1, \\ \Gamma(C_0) = \lambda C_0, \end{array} \right. \left\{ \begin{array}{c} \Gamma(e_1) = e_1, \\ \Gamma(E_1) = \alpha E_0 - \beta F_0, \\ \Gamma(F_1) = -\gamma E_0 + \delta F_0, \\ \Gamma(C_1) = \lambda C_1, \end{array} \right. \tag{42}$$

where

$$\left\{ \begin{array}{l} \alpha = ae^{i\phi}, \\ \beta = \pm be^{i\frac{(\phi+\psi)}{2}}, \\ \gamma = \pm ce^{i\frac{(\phi+\psi)}{2}}, \\ \delta = ae^{i\psi}, \\ \lambda = (\alpha\delta - \beta\gamma) = e^{i(\phi+\psi)}, \end{array} \right. \tag{43}$$

with $a, b, c \geq 0$ fulfilling $a^2 - bc = 1$.
Type II: $\Gamma = I \circ \varphi, \varphi \in Aut_{II}^S(H_1^i)$ is given by

$$\left\{ \begin{array}{c} \Gamma(e_0) = e_1, \\ \Gamma(E_0) = \alpha E_0 + \beta F_0, \\ \Gamma(F_0) = \gamma E_0 + \delta F_0, \\ \Gamma(C_0) = \lambda C_1, \end{array} \right. \left\{ \begin{array}{c} \Gamma(e_1) = e_0, \\ \Gamma(E_1) = -\alpha E_1 + \beta F_1, \\ \Gamma(F_1) = \gamma E_1 - \delta F_1, \\ \Gamma(C_1) = \lambda C_0, \end{array} \right. \tag{44}$$

where

$$\left\{ \begin{array}{l} \alpha = ae^{i\phi}, \\ \beta = \pm ibe^{i\frac{(\phi+\psi)}{2}}, \\ \gamma = \pm ice^{i\frac{(\phi+\psi)}{2}}, \\ \delta = ae^{i\psi}, \\ \lambda = -(\alpha\delta - \beta\gamma) = -e^{i(\phi+\psi)}, \end{array} \right. \tag{45}$$

with $a > 0, b, c \geq 0$ fulfilling $a^2 + bc = 1$.
2) The $*$-operations characterized under (I) above are in fact the Hopf $*$-operations of H_1^i. Indeed they all fulfill $\Delta(\Gamma(a)) = \Gamma \otimes \Gamma(\Delta(a)), a \in H_1^i$, whilst this is the case for none of the $*$-operations (II). One has thus a four-parameter family of Hopf $*$-operations belonging to the same orbit of right action of Hopf homomorphisms. Observe that the $*$-operation I is obtained by making in (I) the choice $\alpha = \delta = 1, \beta = \gamma = 0$ whilst the choice $\alpha = \delta = 0, \beta = -i, \gamma = i$ yields the $*$-operation

$$\left\{ \begin{array}{c} e_0 \to e_0, \\ E_0 \to -iF_1, \\ F_0 \to iE_1, \\ C_0 \to -C_0, \end{array} \right. \left\{ \begin{array}{c} e_1 \to e_1, \\ E_1 \to iF_0, \\ F_1 \to -iE_0, \\ C_1 \to -C_1, \end{array} \right. \left\{ \begin{array}{c} K \to K, \\ E \to iKF, \\ F \to iEK^{-1}. \end{array} \right.$$

Proof:

1) We seek the semi-Hopf $*$-operations as the composition products $I \circ \varphi$, $\varphi \in Aut^S(H_1^i)$, which are involutions. With $*$ indicating complex conjugation, we have

-for $\varphi \in Aut_I^S(H_1^i)$ iteration of 42 will yield the identity operation iff $M\bar{M} = \mathbf{1}$, M the matrix $\begin{pmatrix} \mu & \nu \\ \sigma & \tau \end{pmatrix}$, and $\lambda\bar{\lambda} = 1$. Setting $\mu^* = \alpha, \nu^* = \beta, \sigma^* = \gamma, \tau^* = \delta$, the first condition yields

$$\left\{ \begin{array}{l} 1)\alpha\alpha^* - \beta\gamma^* = 1, \\ 2)\delta\delta^* - \beta\gamma^* = 1, \end{array} \right. \left\{ \begin{array}{l} 3)\alpha\beta^* = \beta\delta^*, \\ 4)\gamma\alpha^* = \delta\gamma^*, \end{array} \right.$$

expressed by 43 (observe that $a \neq 0$ and that in the case $b = c = 0$ the phases of β and γ are arbitrary); the second condition is then automatic,

-for $\varphi \in Aut^S_{II}(H^i_1)$ iteration of 44 will yield the identity operation iff $M\bar{M} = \mathbf{1}$, M the matrix $\begin{pmatrix} \mu & -\nu \\ -\sigma & \tau \end{pmatrix}$, and $\lambda\bar{\lambda} = 1$. Setting $\mu^* = \alpha, \nu^* = \beta, \sigma^* = \gamma, \tau^* = \delta$, the first condition yields

$$\begin{cases} 1)\alpha\alpha^* + \beta\gamma^* = 1, \\ 2)\delta\delta^* + \beta\gamma^* = 1, \end{cases} \begin{cases} 3)\alpha\beta^* + \beta\delta^* = 0, \\ 4)\gamma\alpha^* + \delta\gamma^* = 0, \end{cases}$$

expressed by 45 (observe that $a \neq 0$ and that in the case $b = c = 0$ the phases of β and γ are arbitrary); the second condition is then automatic.

2)Follows from 5.2.2. or can be checked analogously. $\square$

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